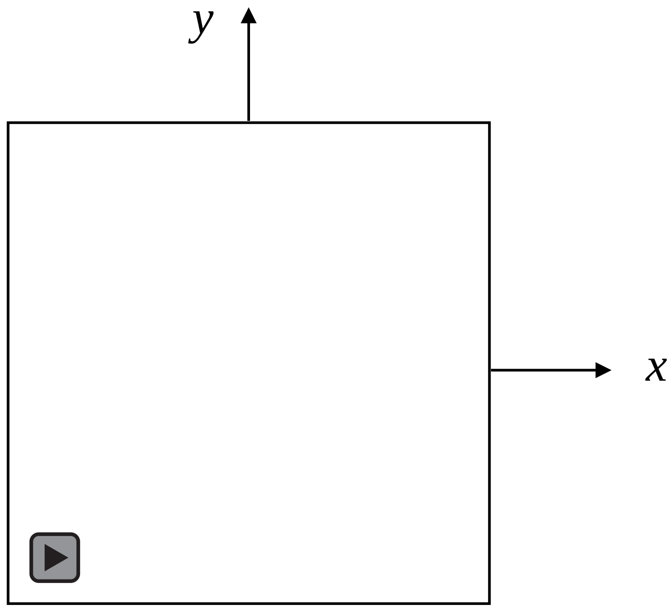
Kreisbewegung



$$\vec{r} = R\cos(\omega t)\,\hat{x} + R\sin(\omega t)\,\hat{y}$$

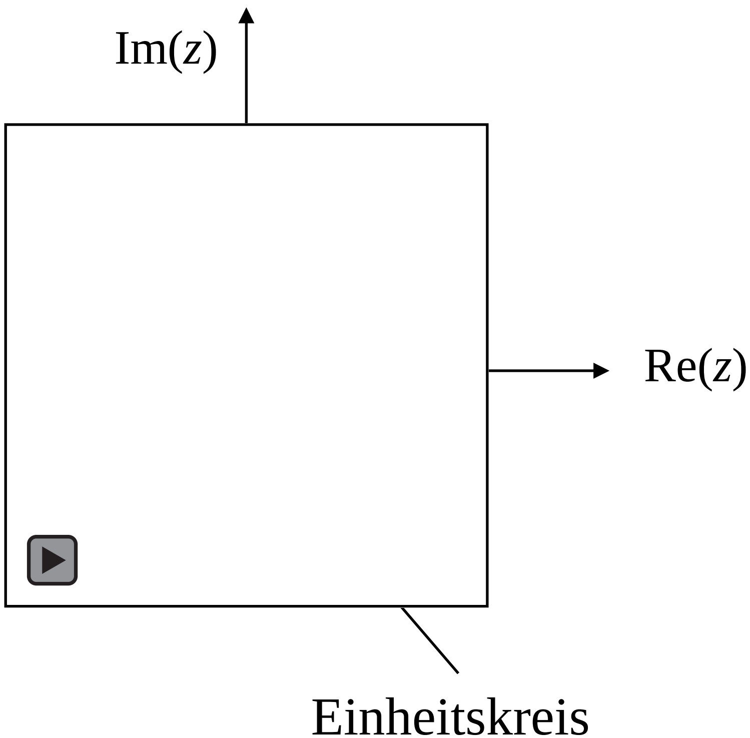
$$\omega = 2\pi f$$

Frequenz [1/s] = [Hz]

Winkelgeschwindigkeit [rad/s]

Euler'sche Formel $e^{i\theta} = \cos\theta + i\sin\theta$

$$i^2 = -1 = \left(j^2\right)$$



$$e^{i\omega t} = \cos\left(\omega t\right) + i\sin\left(\omega t\right)$$

ω = Kreisfrequenz

$$\left|e^{i\theta}\right| = \sqrt{e^{-i\theta}e^{i\theta}} = \sqrt{e^{0}} = 1 = \sqrt{\left(\cos\theta - i\sin\theta\right)\left(\cos\theta + i\sin\theta\right)} = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

Lineare gewöhnliche Differentialgleichung

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0 \qquad (1)$$

$$m\frac{d^2z}{dt^2}+b\frac{dz}{dt}+kz=0 \qquad (2)$$

$$z(t)=x_1(t)+ix_2(t)$$

$$m\frac{d^2x_1}{dt^2}+b\frac{dx_1}{dt}+kx_1+im\frac{d^2x_2}{dt^2}+ib\frac{dx_2}{dt}+ikx_2=0$$

Wenn z als Lösung zu Glchg. (2) gefunden wurde, sind x_1 als auch x_2 Lösungen zu Glchg. (1)

Lineare gewöhnliche Differentialgleichung

$$m\frac{d^2z}{dt^2}+b\frac{dz}{dt}+kz=0$$

Lösung: $z(t)=x_1(t)+ix_2(t)=Ce^{\lambda t}$

Konstant

$$m\lambda^2Ce^{\lambda t}+b\lambda Ce^{\lambda t}+kCe^{\lambda t}=0$$

$$m\lambda^2+b\lambda+k=0$$

Lineare gewöhnliche Differentialgleichung

$$m\lambda^2 + b\lambda + k = 0$$

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4mk}}{2m} \qquad \lambda_2 = \frac{-b - \sqrt{b^2 - 4mk}}{2m}$$

Lösung: $$z(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

Anfangsbedingungen:

$$z(0) = C_1 + C_2$$

$$\frac{dz}{dt}(0) = \lambda_1 C_1 + \lambda_2 C_2$$

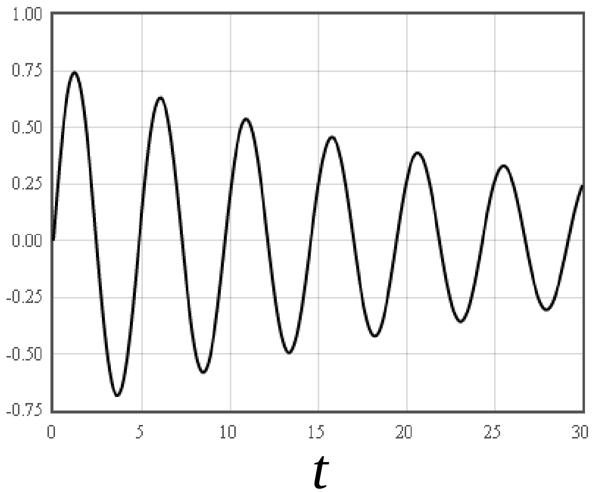
$$C_1 = \frac{\lambda_2 z(0) - \frac{dz}{dt}(0)}{\lambda_2 - \lambda_1} \qquad C_2 = \frac{\lambda_1 z(0) - \frac{dz}{dt}(0)}{\lambda_1 - \lambda_2}$$

$b^2 < 4km$ Schwingfall

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0$$

Lösung: $z(t)=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}$

$x=\mathrm{Re}(z)$



$$\lambda_{1,2}=\frac{-b\pm\sqrt{b^2-4mk}}{2m}$$

$$\tau=\frac{2m}{b}$$

$$\lambda_{1,2}=\frac{-b\pm\sqrt{-1\left(4mk-b^2\right)}}{2m}$$

$$\omega_0=\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}$$

$$\lambda_{1,2}=\frac{-b\pm i\sqrt{4mk-b^2}}{2m}=\frac{-1}{\tau}\pm i\omega_0$$

$b^2 < 4km$ Schwingfall

$$z(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

$$\lambda_{1,2} = \frac{-1}{\tau} \pm i\omega_0$$

$$e^{(a+b)} = e^a e^b$$

$$z(t) = C_1 e^{-t/\tau} e^{i\omega_0 t} + C_2 e^{-t/\tau} e^{-i\omega_0 t}$$

$$e^{i\omega_0 t} = \cos\omega_0 t + i\sin\omega_0 t$$

$$z(t) = C_1 e^{-t/\tau}\left(\cos\omega_0 t + i\sin\omega_0 t\right) + C_2 e^{-t/\tau}\left(\cos\omega_0 t - i\sin\omega_0 t\right)$$

$$x_1(t) = \mathrm{Re}\left(z(t)\right)$$

Differentialgleichungen zweiter Ordnung

Lösung Differentialgleichungen zweiter Ordnung

$$a\,\frac{d^2x}{dt^2} + b\,\frac{dx}{dt} + cx = d,$$

$a =$ 1

$b =$ 1

$c =$ 1

$d =$ 1

Anfangsbedingungen:

$x(t_0) =$ 1

$\frac{dx}{dt}(t_0) =$ 1

$t_0 =$ 0

Lösung

Lineare gewöhnliche Differentialgleichung

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0$$

Lineare Differentialgleichung zweiter Ordnung

Lösung: $x = Ce^{\lambda t}$

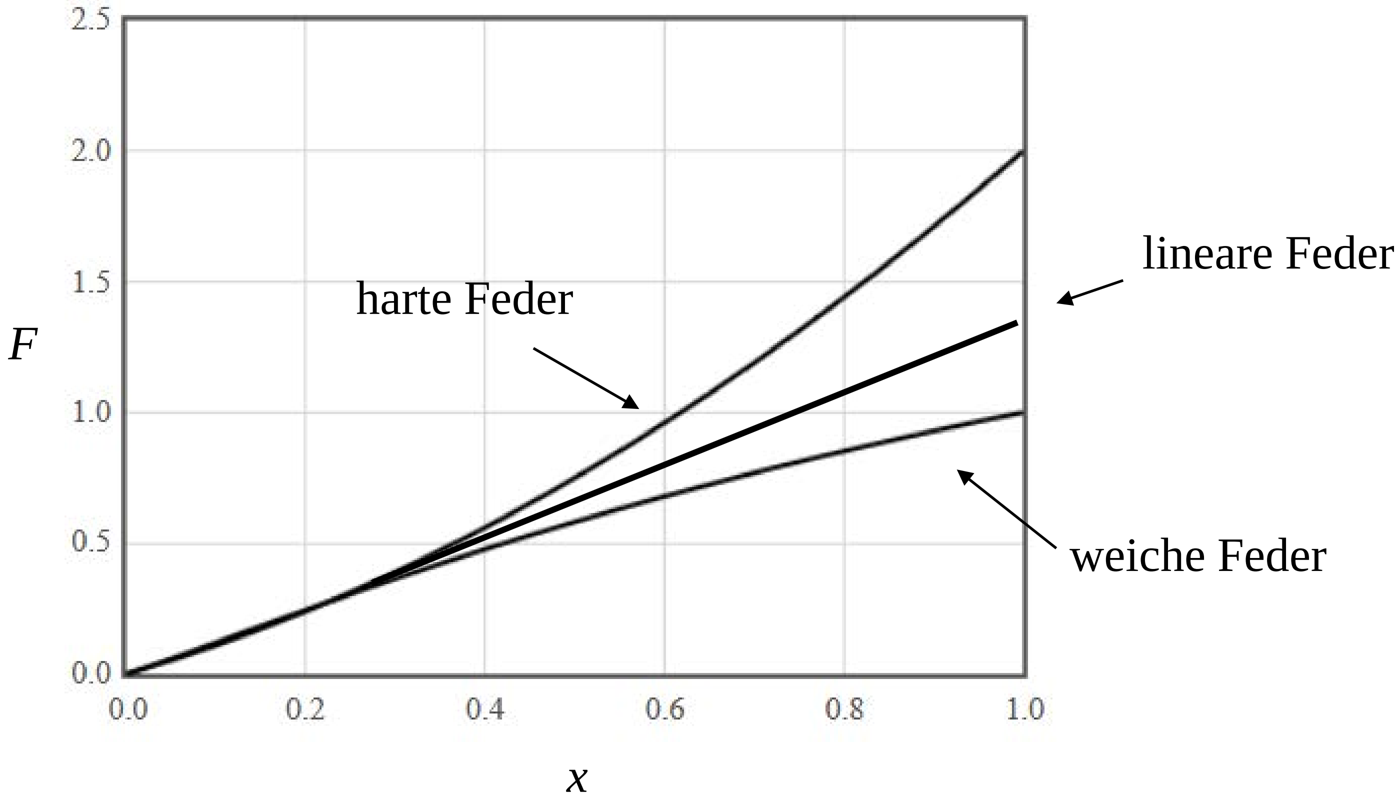
Nichtlineare Differentialgleichung zweiter Ordnung

$$m\left(\frac{d^2x}{dt^2}\right)^2+b\frac{dx}{dt}+kx=0$$

$$m\frac{d^2x}{dt^2}+b\left(\frac{dx}{dt}\right)^2+kx=0$$

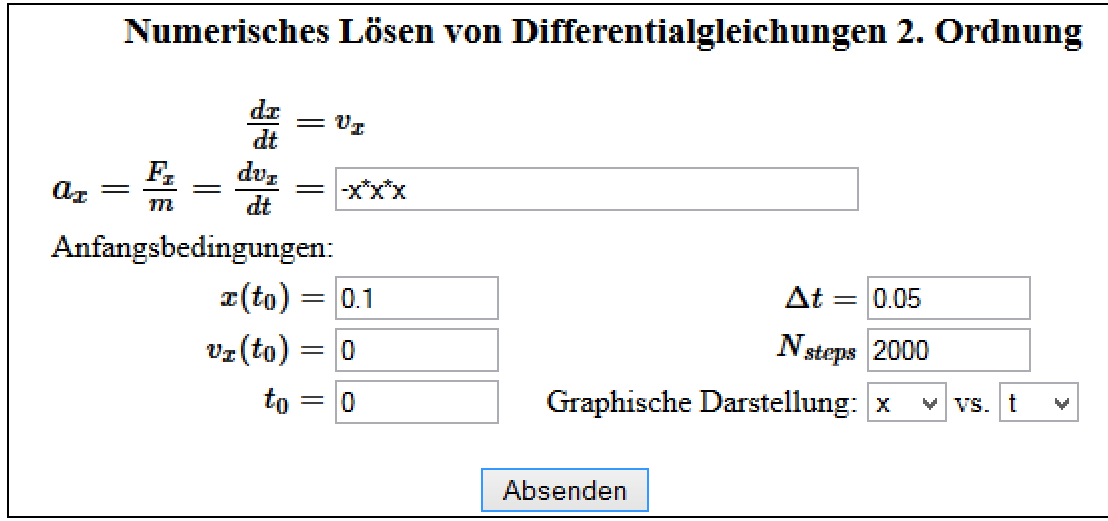
$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx^3=0$$

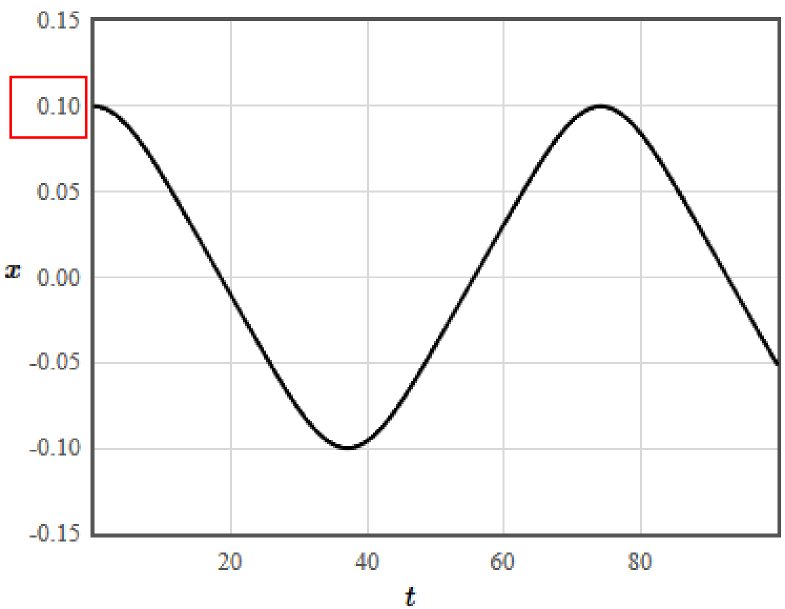
harte/weiche Feder

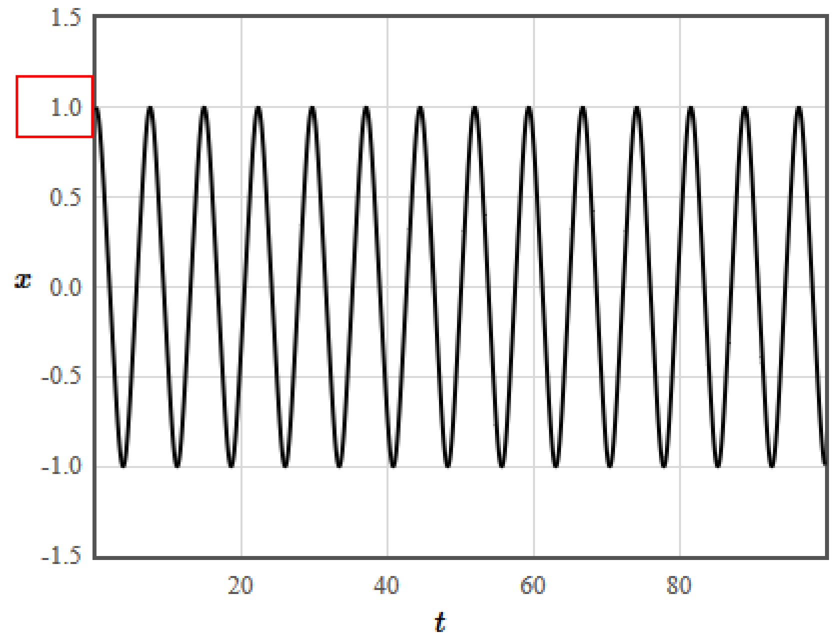


harte Feder

$$m\frac{d^2x}{dt^2}+kx^3=0$$

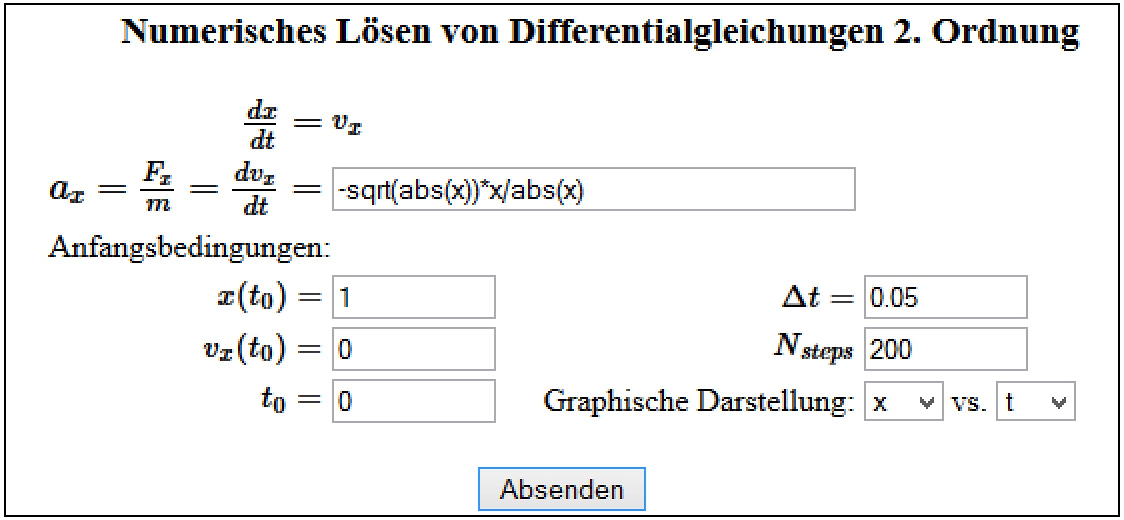


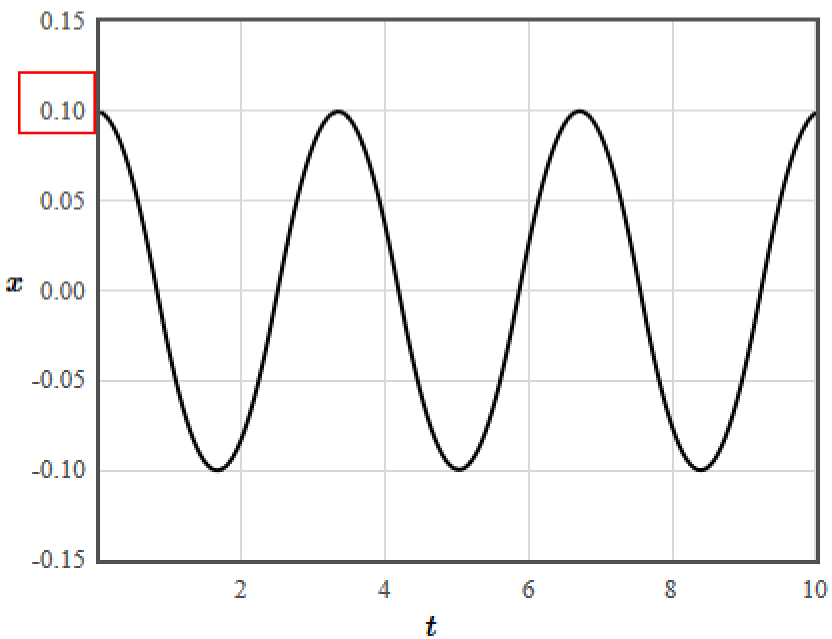


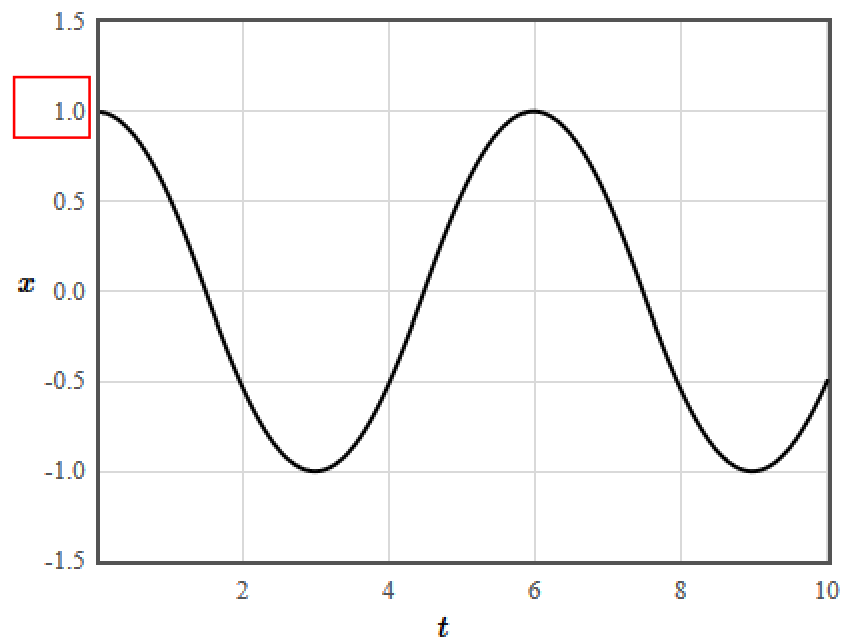


weiche Feder

$$m\frac{d^2x}{dt^2}+k\,\mathrm{sgn}(x)\sqrt{|x|}=0$$







Pendel (weiche Feder)

$$m\frac{d^2x}{dt^2} \approx -\frac{mg}{\ell}x$$

$$\omega_0 = \sqrt{\frac{g}{\ell}}$$

Numerisches Lösen von Differentialgleichungen 2. Ordnung

$$\frac{dx}{dt} = v_x$$

$a_x = \frac{F_x}{m} = \frac{dv_x}{dt} =$ -sin(x)

Anfangsbedingungen:

$x(t_0) =$ 0.1

$v_x(t_0) =$ 0

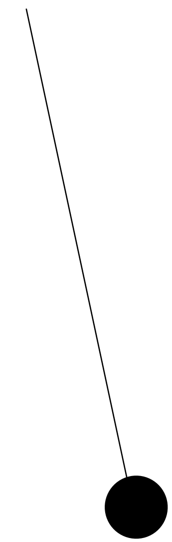
$t_0 =$ 0

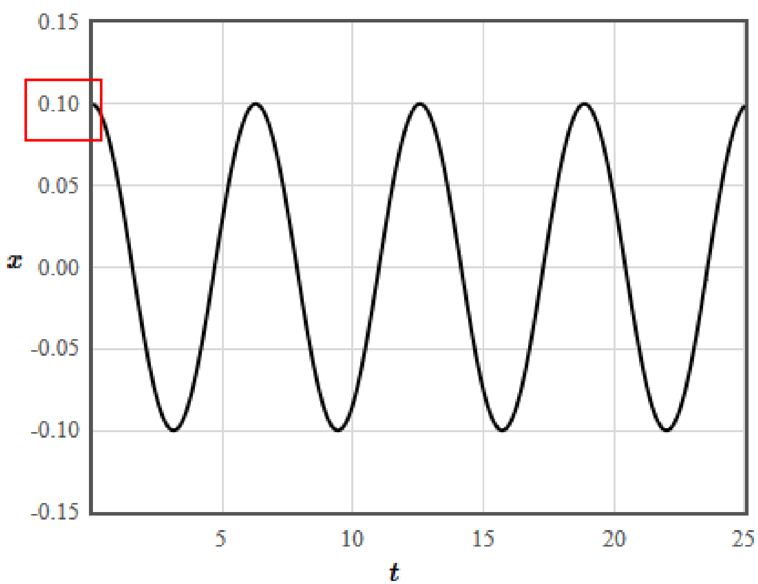
$\Delta t =$ 0.05

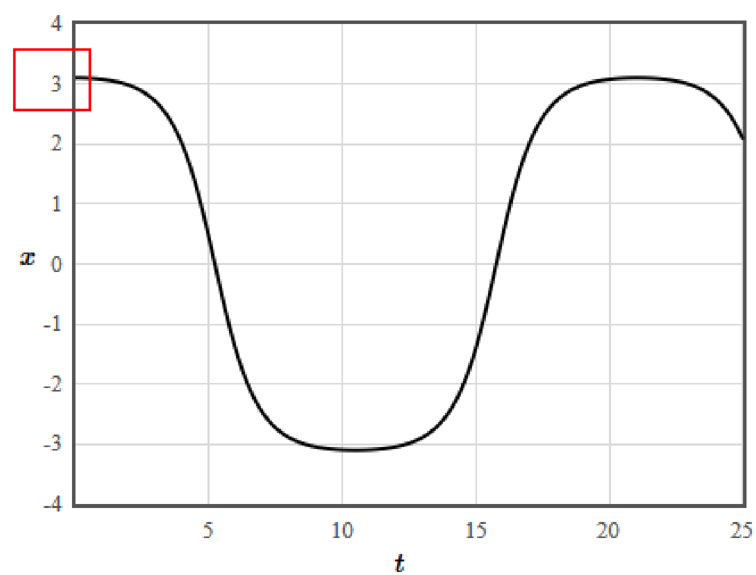
N_{steps} 500

Graphische Darstellung: x vs. t

Absenden







Relaxationsoszillator

Numerisches Lösen von Differentialgleichungen 2. Ordnung

$$\frac{dx}{dt} = v_x$$

$$a_x = \frac{F_x}{m} = \frac{dv_x}{dt} = $$ (1-x*x)*vx-x

Anfangsbedingungen:

$x(t_0) =$ 0 $\Delta t =$ 0.05

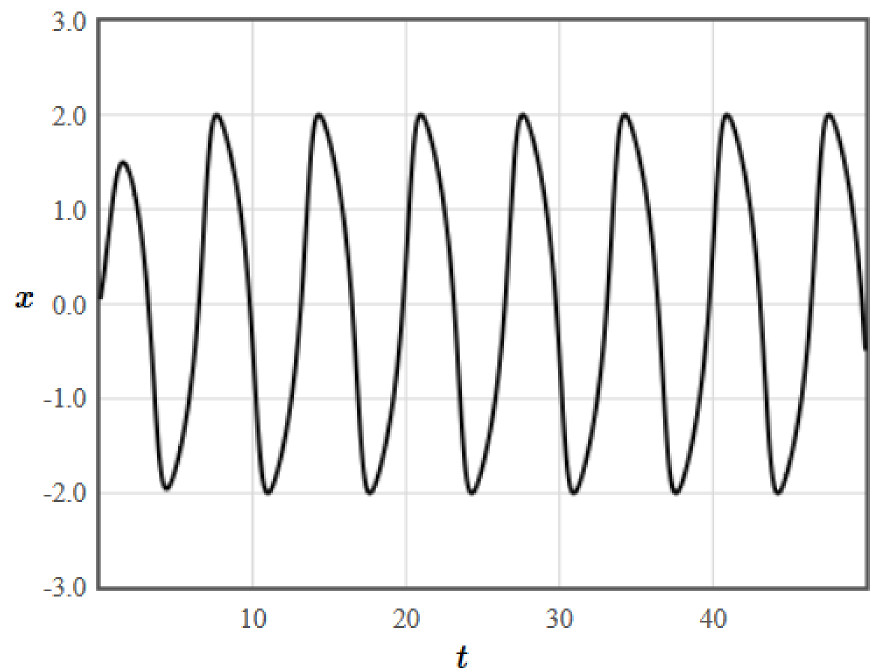
$v_x(t_0) =$ 1 N_{steps} 1000

$t_0 =$ 0 Graphische Darstellung: x vs. t

Absenden

Van-der-Pol-System

$$\frac{d^2x}{dt^2} - \mu\left(1 - x^2\right)\frac{dx}{dt} + x = 0$$



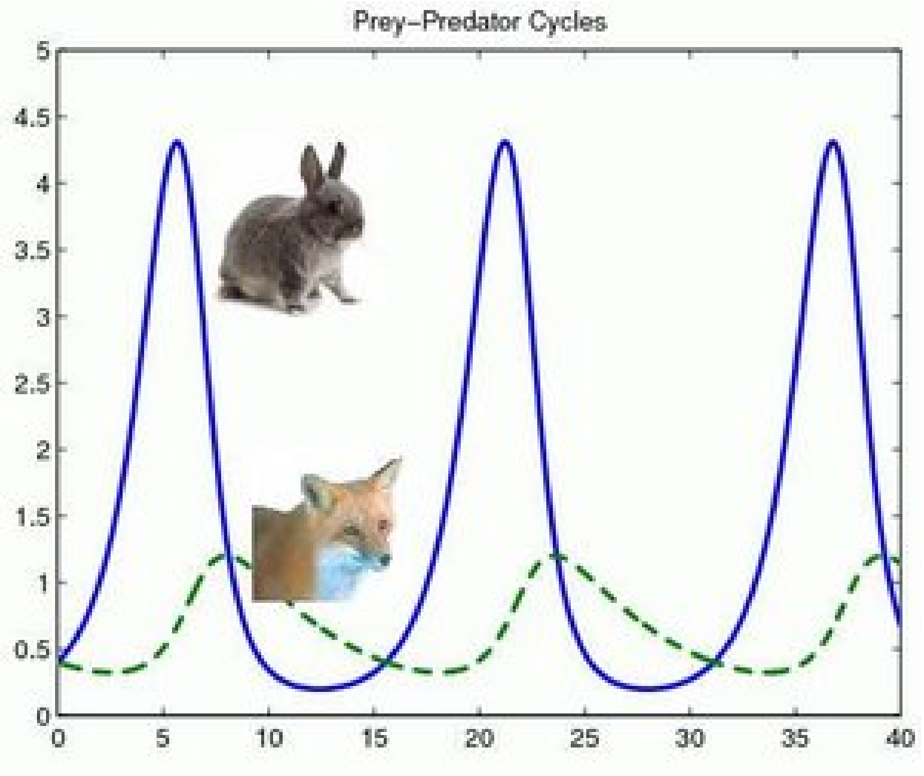
negativen Reibungs für kleine x

Räuber-Beute-Gleichungen

$$\frac{dx_1}{dt} = \left(b - px_2\right)x_1$$

$$\frac{dx_2}{dt} = \left(rx_1 - d\right)x_2$$

Lotka-Volterra Gleichungen



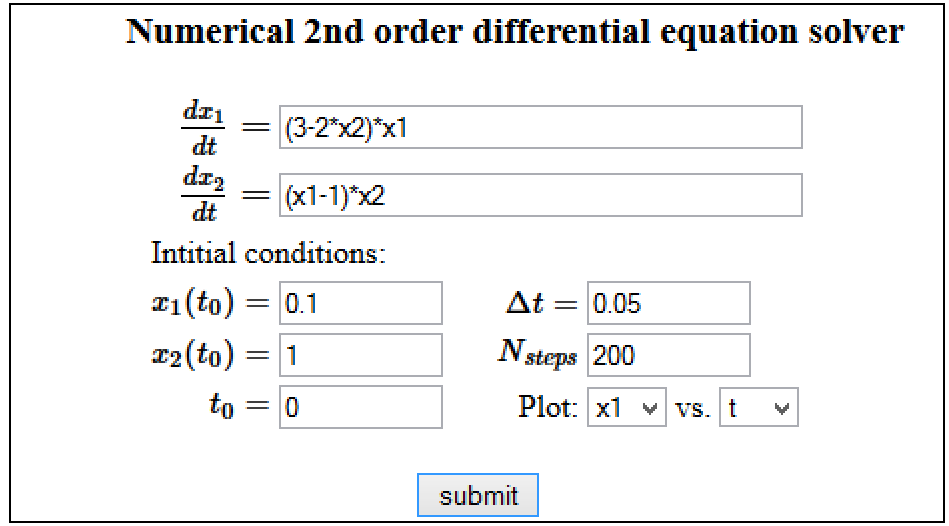
http://www.scholarpedia.org/article/Predator-prey_model

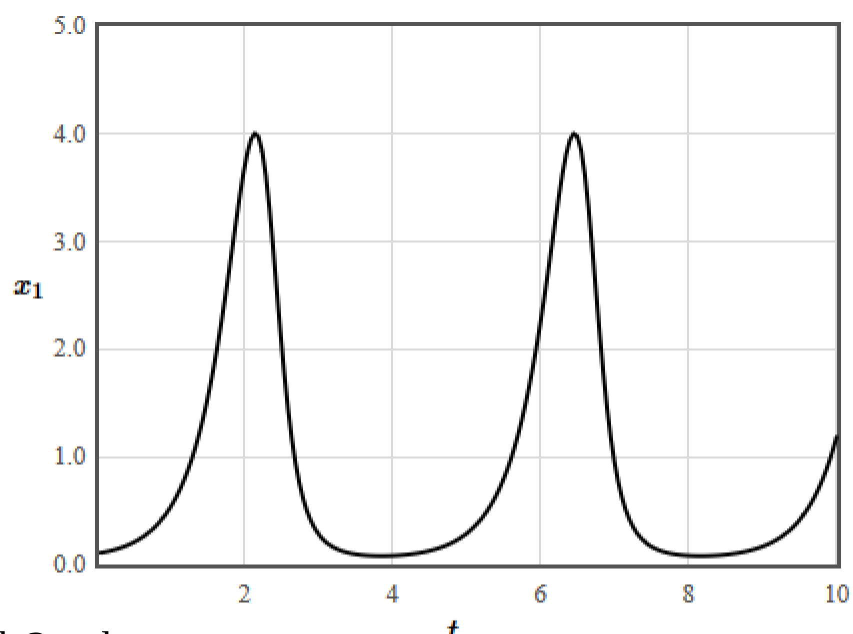
Räuber-Beute-Gleichungen

$$\frac{dx_1}{dt} = \left(b - px_2\right) x_1$$

$$\frac{dx_2}{dt} = \left(rx_1 - d\right) x_2$$

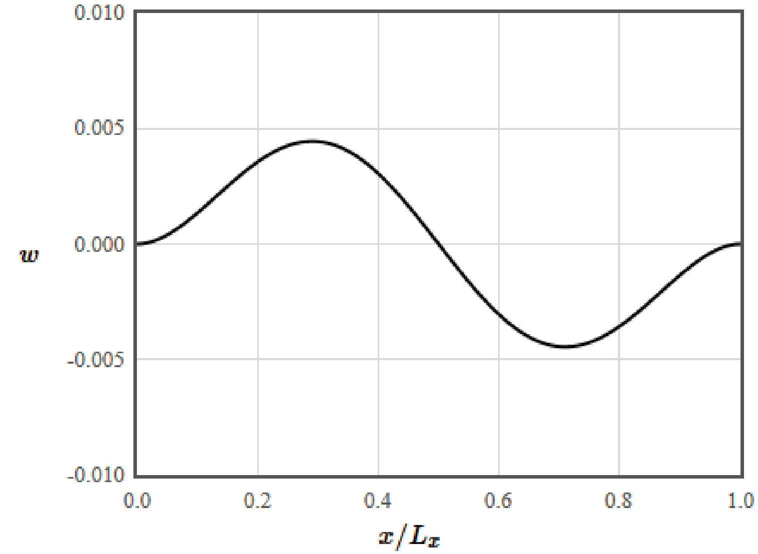
Lotka-Volterra Gleichungen





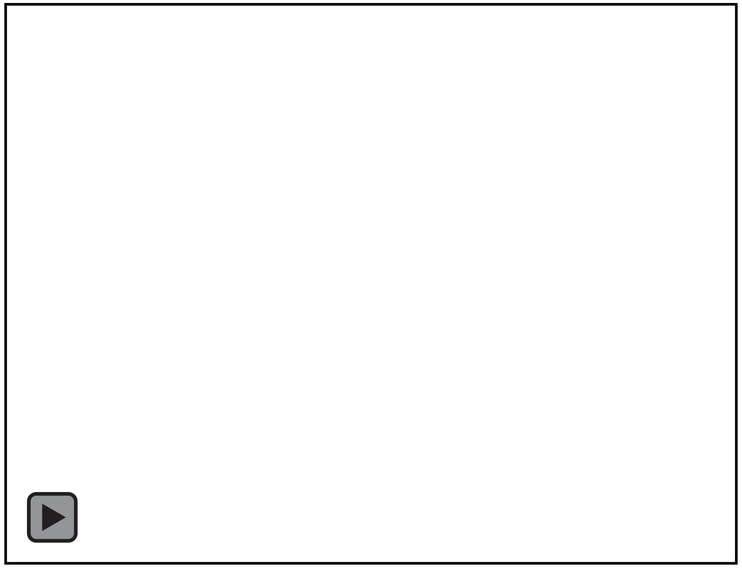
http://lamp.tu-graz.ac.at/~hadley/physikm/apps/ode2n.php

Balken und Cantilevers



Euler-Lagrange Gleichung

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = -\mu\frac{\partial^2 w}{\partial t^2} + q(x).$$

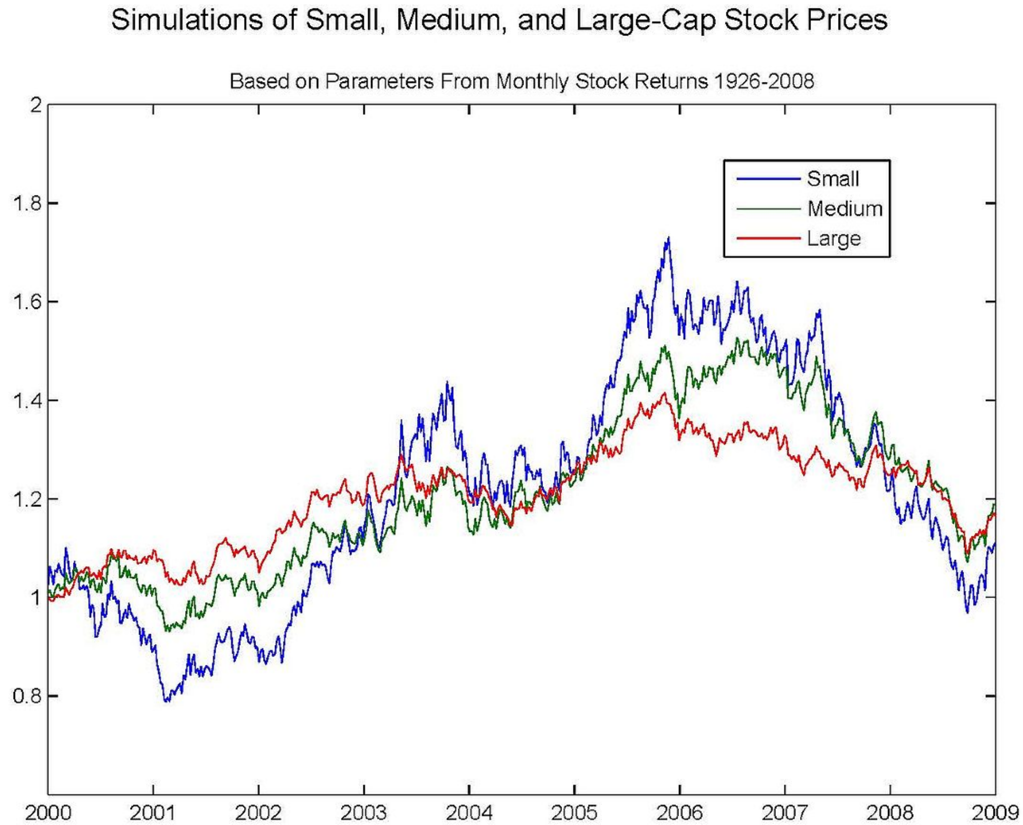


http://lampx.tugraz.at/~hadley/memm/mechanics/beams.php

Black-Scholes-Modell

$$\frac{\partial V}{\partial t} + r\,S\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2 V}{\partial S^2} = r\,V$$

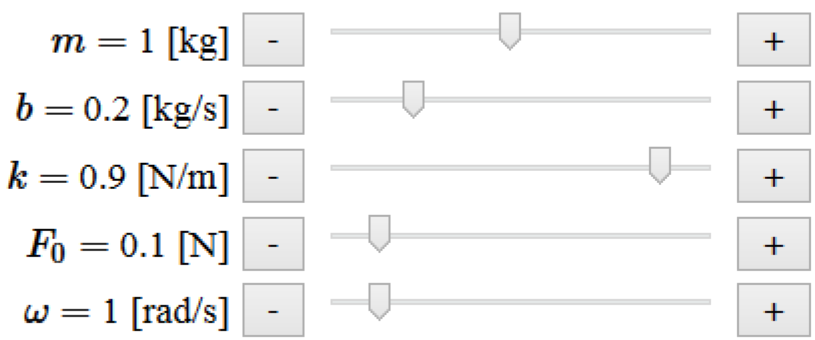
V = Wert einer Option
S = Basiswert
r = Zinssatz
σ = Volatilität
t = Zeit

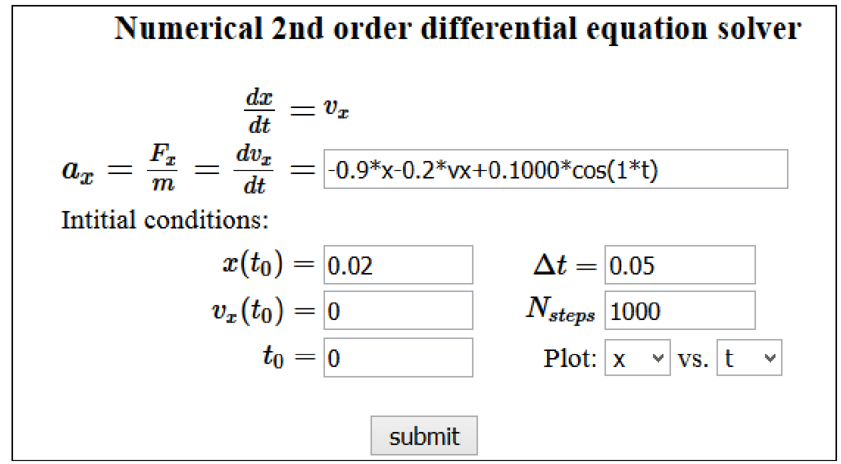


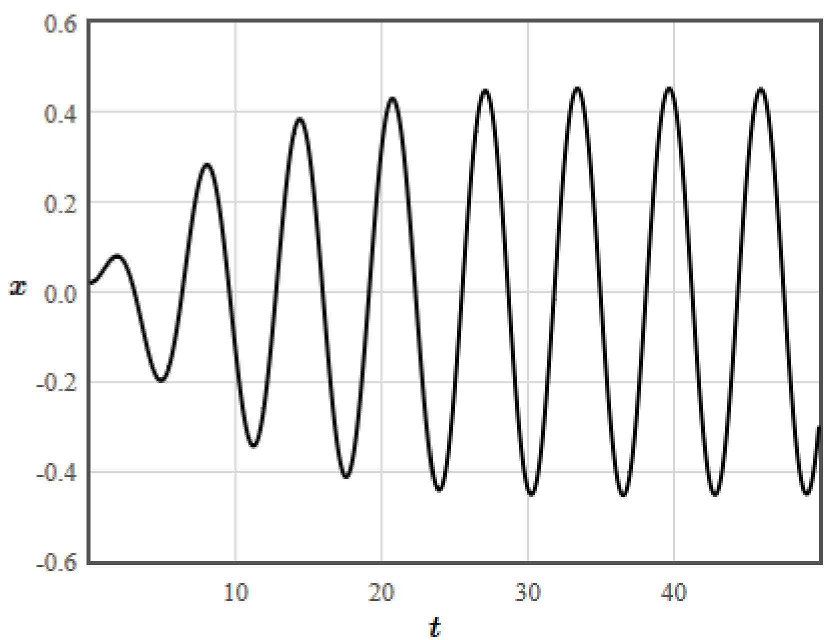
http://de.wikipedia.org/wiki/Black-Scholes-Modell

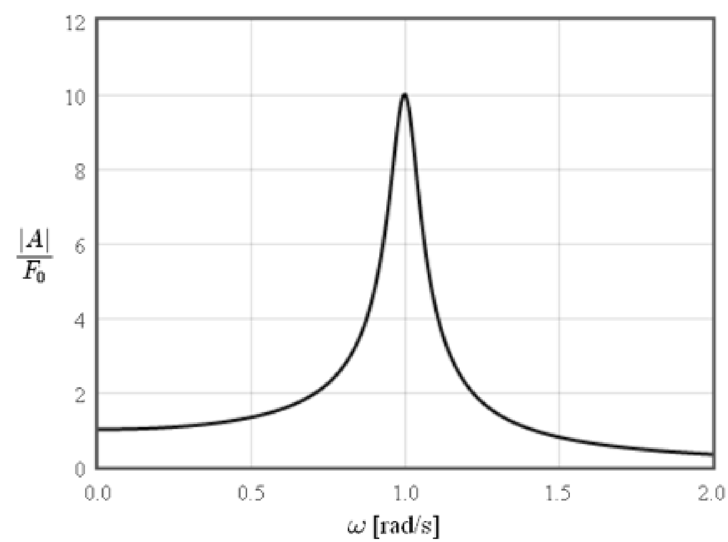
Resonaz

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \cos(\omega t).$$









Resonaz

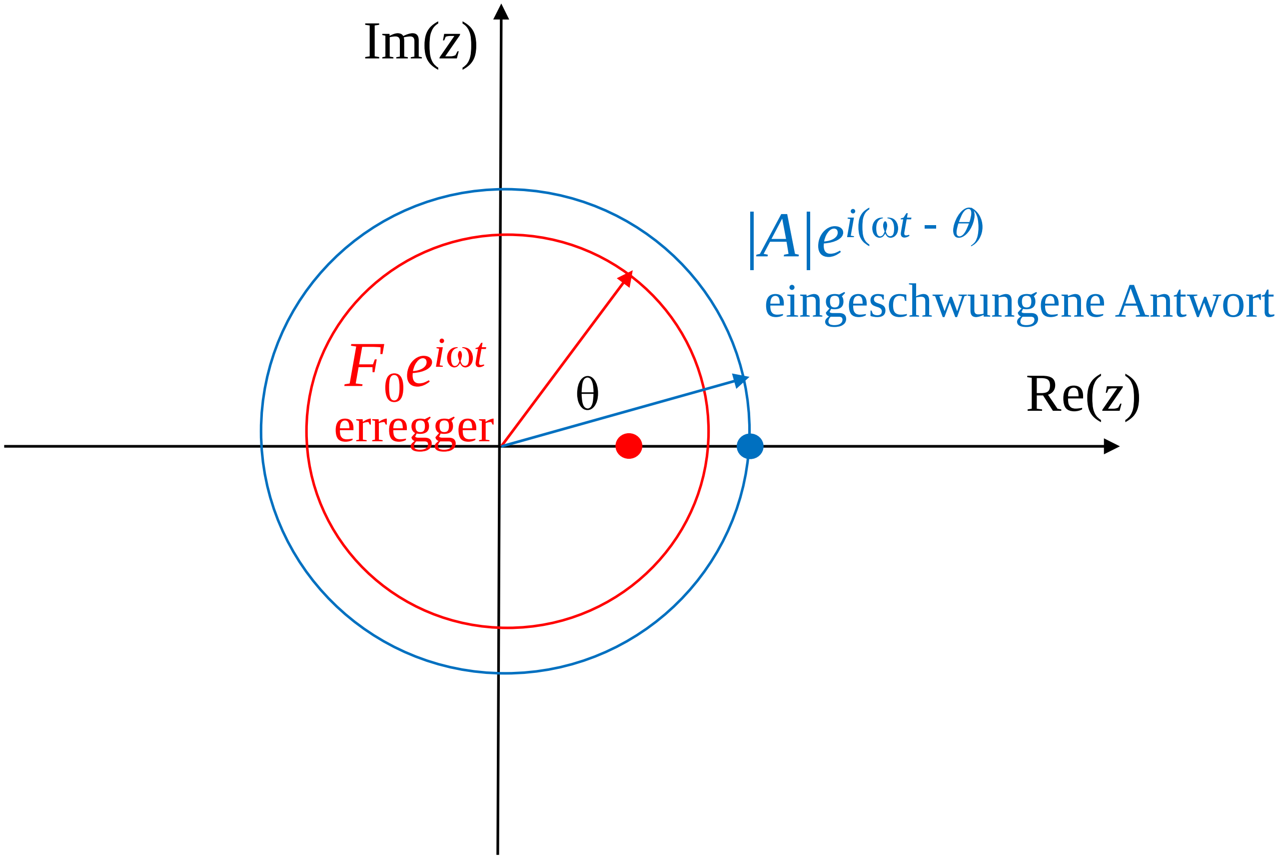
$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=F_0\cos(\omega t)$$

$$m\frac{d^2z}{dt^2}+b\frac{dz}{dt}+kz=F_0\exp(i\omega t)$$

$$z=x_1+ix_2$$

$$m\frac{d^2x_1}{dt^2}+b\frac{dx_1}{dt}+kx_1=F_0\cos(\omega t) \qquad m\frac{d^2x_2}{dt^2}+b\frac{dx_2}{dt}+kx_2=F_0\sin(\omega t)$$

Resonaz



Im(z)
$|A|e^{i(\omega t\ -\ \theta)}$
eingeschwungene Antwort
$F_0 e^{i\omega t}$
erregger
θ
Re(z)

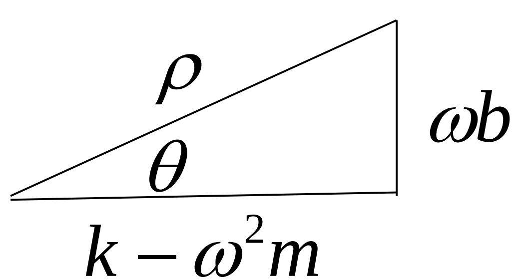
Resonaz

$$m\frac{d^2z}{dt^2}+b\frac{dz}{dt}+kz=F_0\exp\left(i\omega t\right)$$

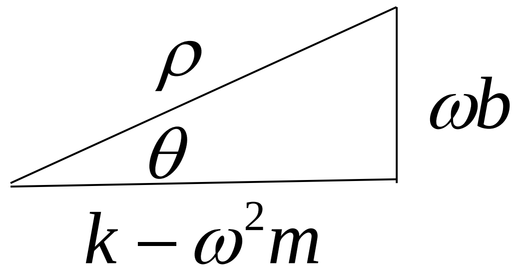
$$z=A\exp\left(i\omega t\right)$$

$$\left(-\omega^2 m+i\omega b+k\right)A=\rho e^{i\theta}A=F_0$$

$$-\omega^2 m+i\omega b+k=\rho e^{i\theta}=\rho\cos\theta+i\rho\sin\theta$$



Resonaz



$$\rho = \sqrt{\left(k - m\omega^2\right)^2 + \omega^2 b^2}$$

$$\tan\theta = \frac{\omega b}{k - m\omega^2}$$

$$A = \frac{F_0}{\rho} e^{-i\theta}$$

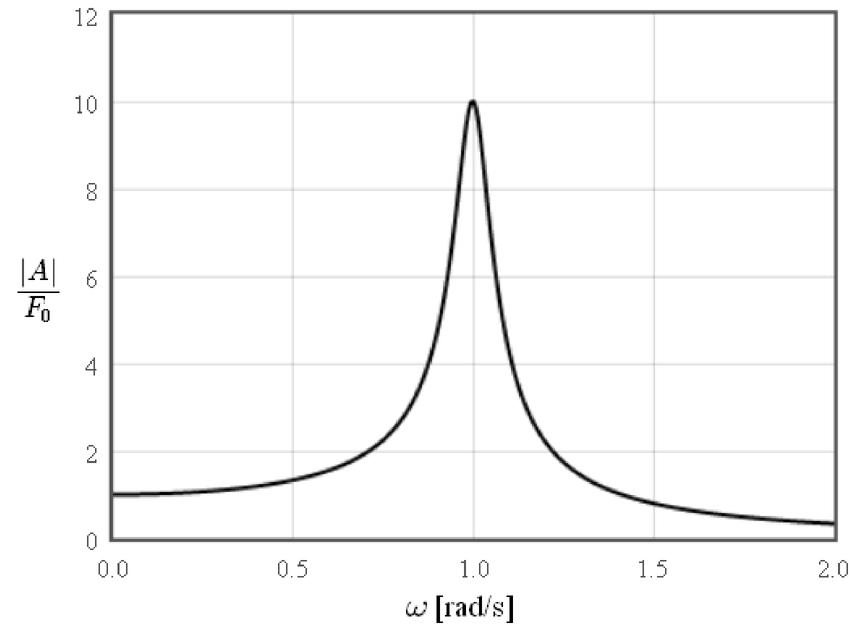
$$z = Ae^{i\omega t} = \frac{F_0}{\rho} e^{i\omega t} e^{-i\theta}$$

$$x_1 = \mathrm{Re}(z) = \frac{F_0}{\rho}\cos(\omega t - \theta)$$

$m =$ 1 [kg] $b =$ 0.1 [N s/m] $k =$ 1 [N/m] Plot

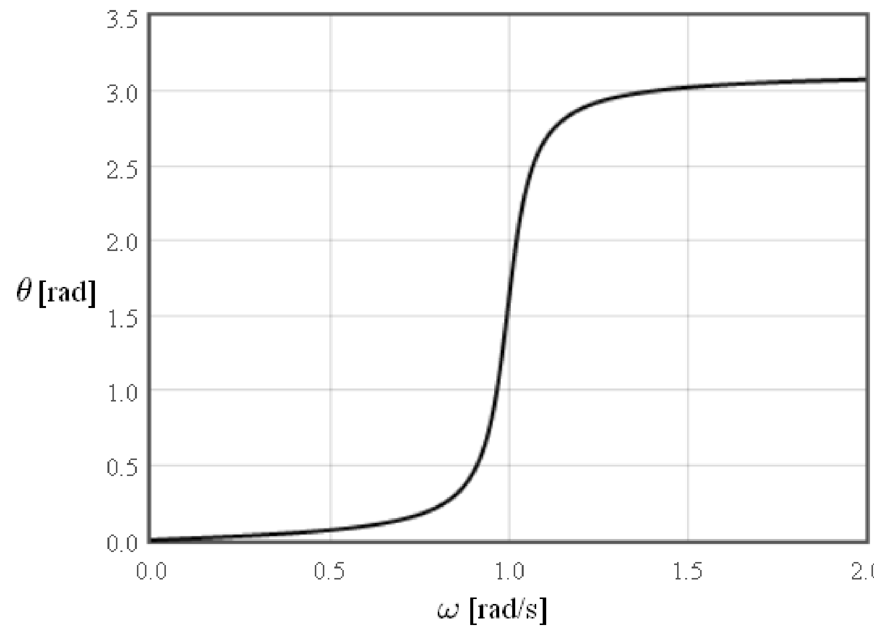
$Q = \frac{\sqrt{mk}}{b} =$ 10

$$\frac{|A|}{F_0} = \frac{1}{\sqrt{\left(k - m\omega^2\right)^2 + \omega^2 b^2}}$$

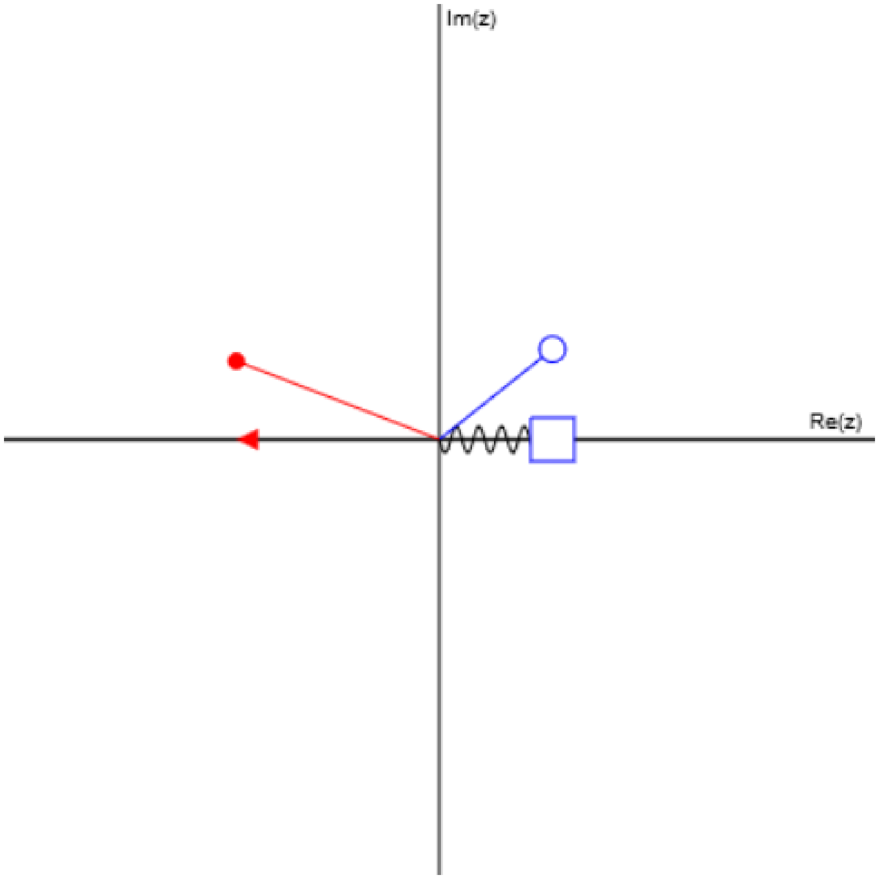


$$A = \frac{F_0}{\rho} e^{-i\theta}$$

$$\tan\theta = \frac{\omega b}{k - m\omega^2}$$



Resonaz



m =4 [kg]

b =1 [N s/m]

k =6 [N/m]

F_0 =1 [N]

ω =1.3 [rad/s]

$$\omega_0 = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = 1.22 \text{ [rad/s]} = 0.194 \text{ [Hz]}$$

$$\theta = \text{atan}\left(\frac{\omega b}{k - m\omega^2}\right) = 2.10 \text{ [rad]} = 120 \text{ [deg]}$$

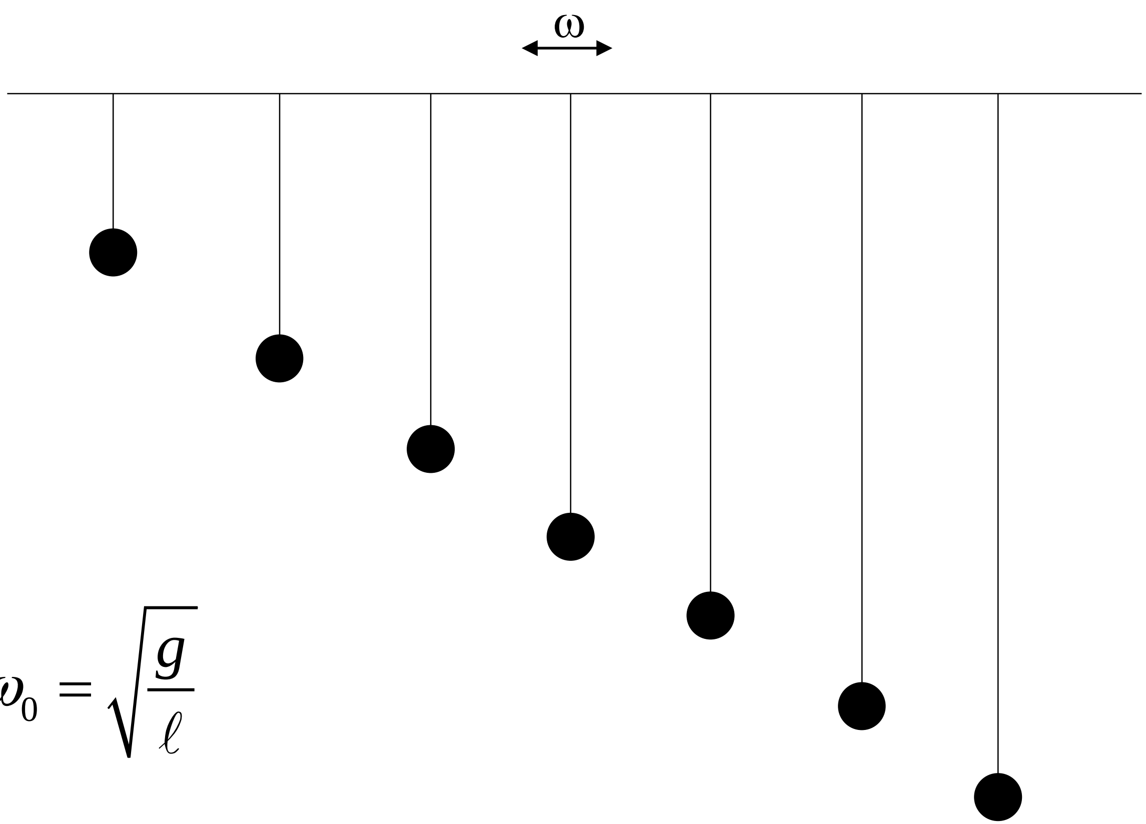
$$A = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \omega^2 b^2}} = 0.664 \text{ [m]}$$

$$Q = \frac{\sqrt{mk}}{b} = 4.90$$

$F_0 e^{i\omega t}$ anzeigen: ☑ $Ae^{i(\omega t - \theta)}$ anzeigen: ☑

z anzeigen: ☐ x_2 anzeigen: ☐

Resonaz

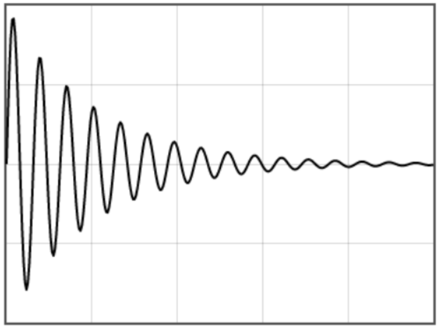


$$\omega_0 = \sqrt{\frac{g}{\ell}}$$

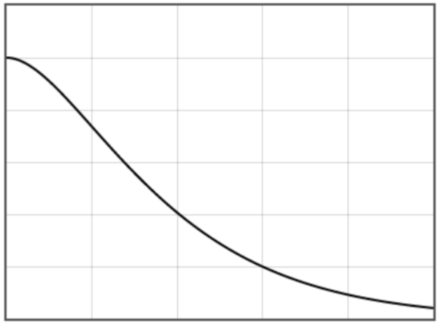
Gütefaktor

$$Q = \frac{\pi\tau}{T}$$

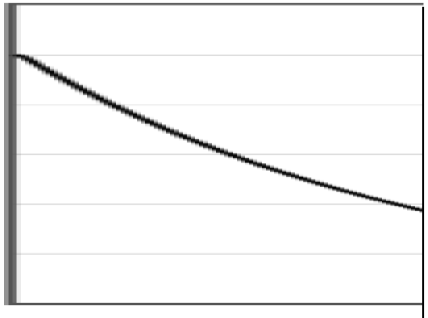
Schwingfall $Q > \frac{1}{2}$



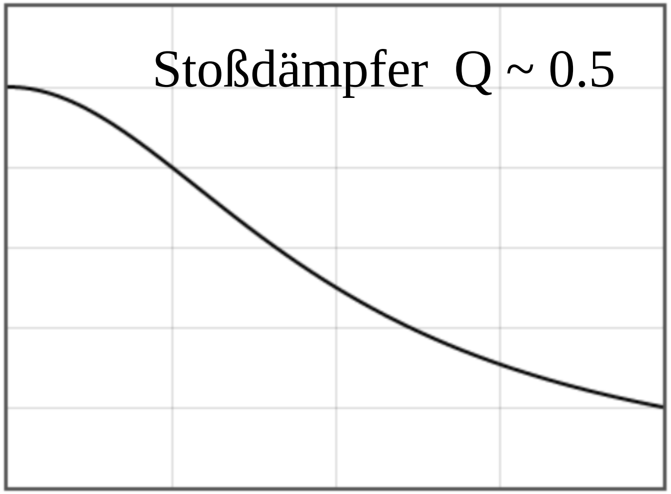
aperiodischer Grenzfall $Q = \frac{1}{2}$

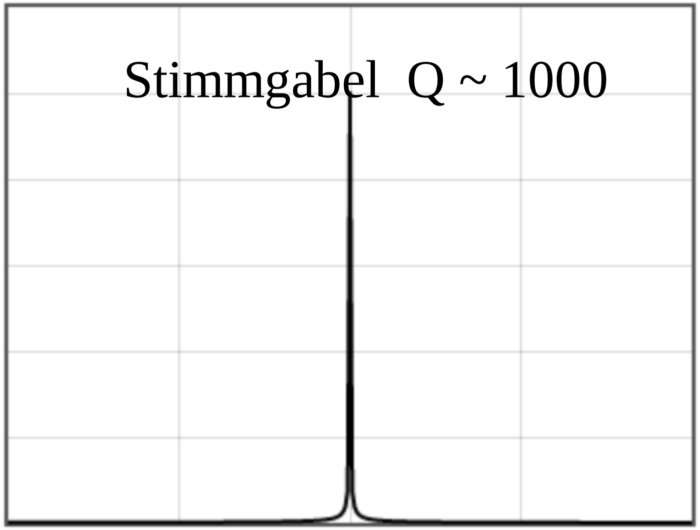


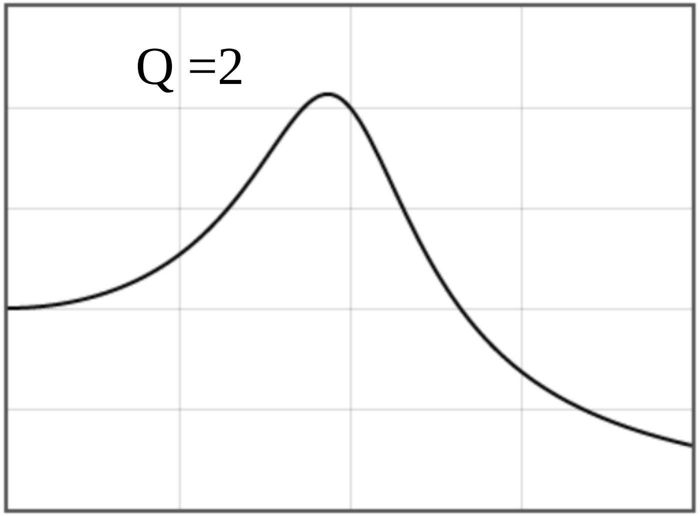
Kriechfall $Q < \frac{1}{2}$

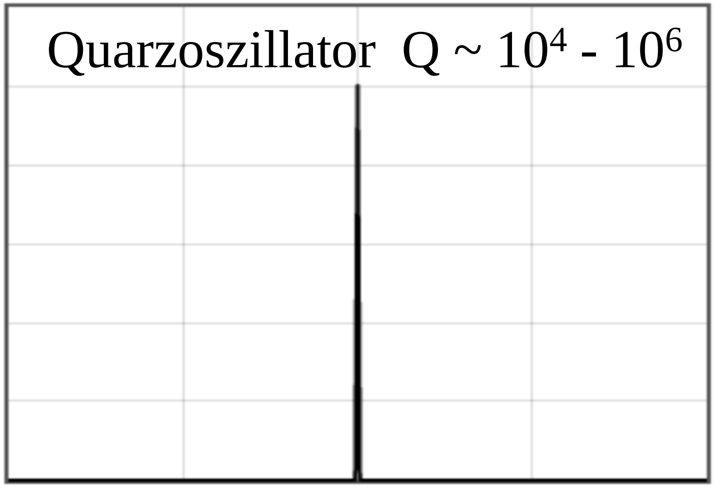


Gütefaktor


Stoßdämpfer Q ~ 0.5


Stimmgabel Q ~ 1000


Q =2


Quarzoszillator Q ~ 10^4 - 10^6

Parametrisch erregte Schwingungen

Numerical 2nd order differential equation solver

$$\frac{dx}{dt} = v_x$$

$$a_x = \frac{F_x}{m} = \frac{dv_x}{dt} = $$ -0.2000*vx-9.81*x/(0.5*(1-0.4*cos(8.3*t)))

Intitial conditions:

$x(t_0) =$ 0.1 $\Delta t =$ 0.05

$v_x(t_0) =$ 0 N_{steps} 1000

$t_0 =$ 0 Plot: x vs. t

submit

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \frac{mg}{l(1 - A\cos(\omega t))}x = 0.$$

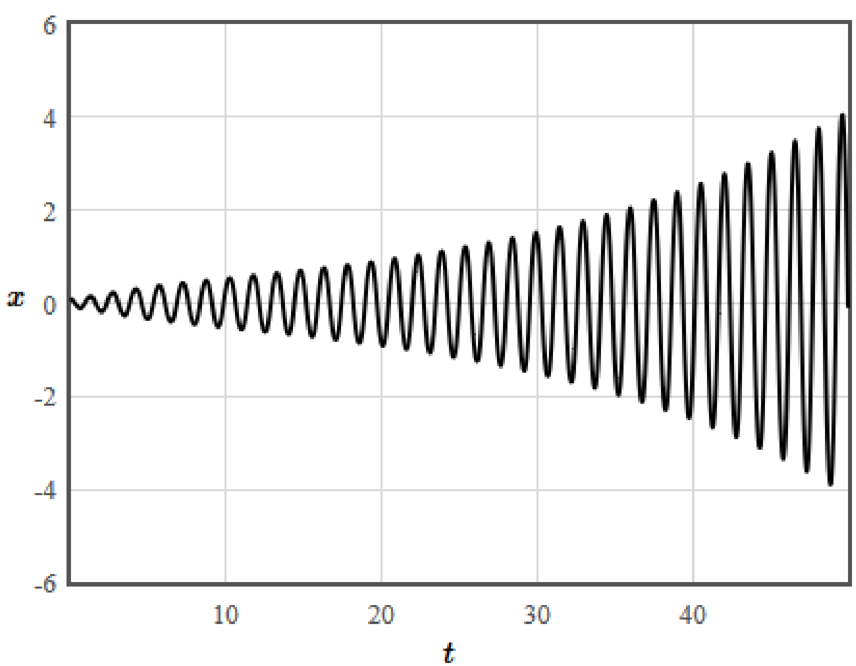
$m = 1$ [kg] - +

$b = 0.2$ [kg/s] - +

$l = 0.5$ [N/m] - +

$A = 0.4$ [N] - +

$\omega = 8.3$ [rad/s] - +



Kind auf einer Schaukel