

19. Schwingungen

▼ Aufgaben

 2.2 Eine Kugel wird in einem Winkel θ geworfen

 2.3 Position \rightarrow Kraft

 8.1 Magnetisches Feld

 8.2 Schraubenförmige Bewegung eines geladenen Teilchens

 9.1 Oszillationen eines Masse-Feder Systems

 9.2 Q-Faktor

Resonanz

Numerical 2nd order differential equation solver

$$\frac{dx}{dt} = vx$$

$$\frac{dv}{dt} = -0.2*vx-3*x+\sin(1*t)$$

Initial conditions:

$$x(t_0) = 0$$

$$\Delta t = 0.05$$

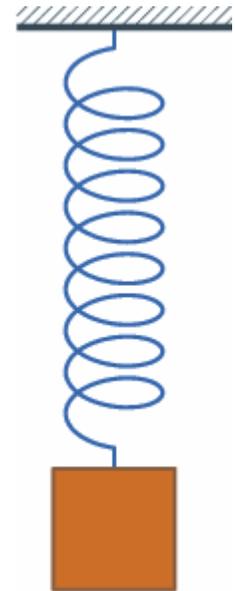
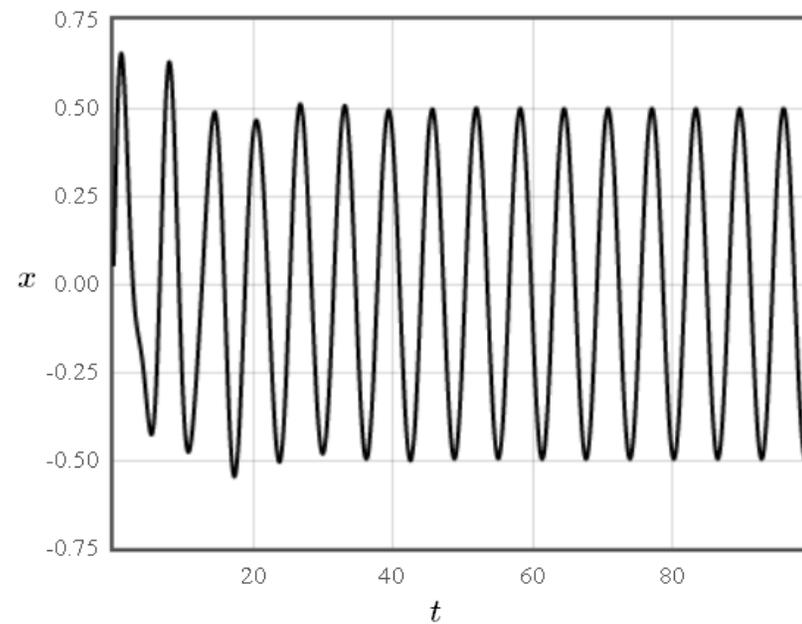
$$v_x(t_0) = 1$$

$$N_{steps} = 2000$$

$$t_0 = 0$$

Plot: x vs. t

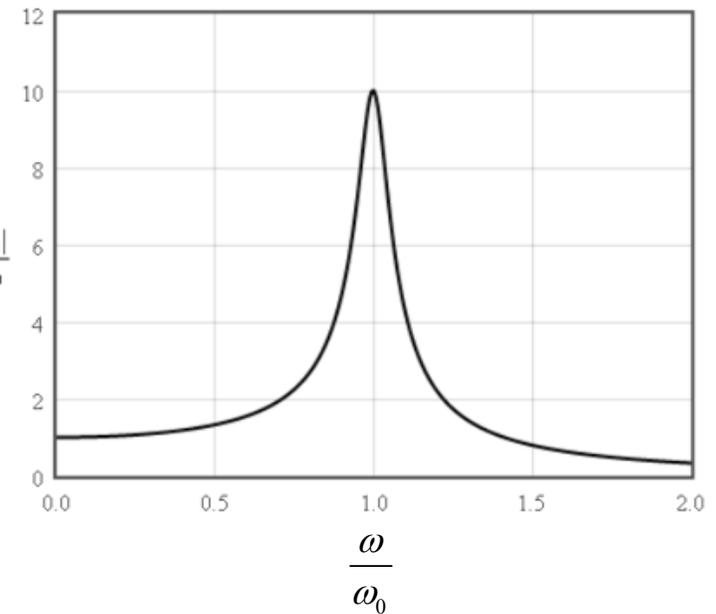
submit



Resonanz

$$ma = F$$

$$m \frac{d^2 x}{dt^2} = -b \frac{dx}{dt} - kx + F_0 \cos(\omega t) \frac{|A|}{F_0}$$



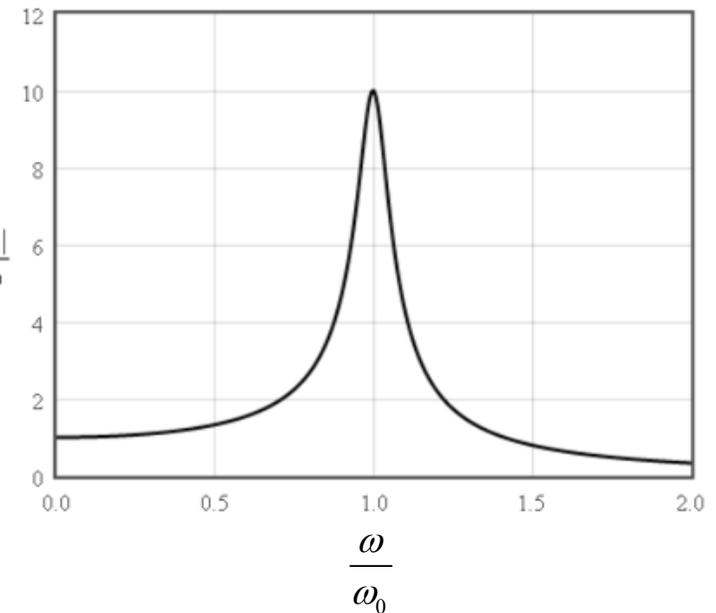
Resonanz

$$ma = F$$

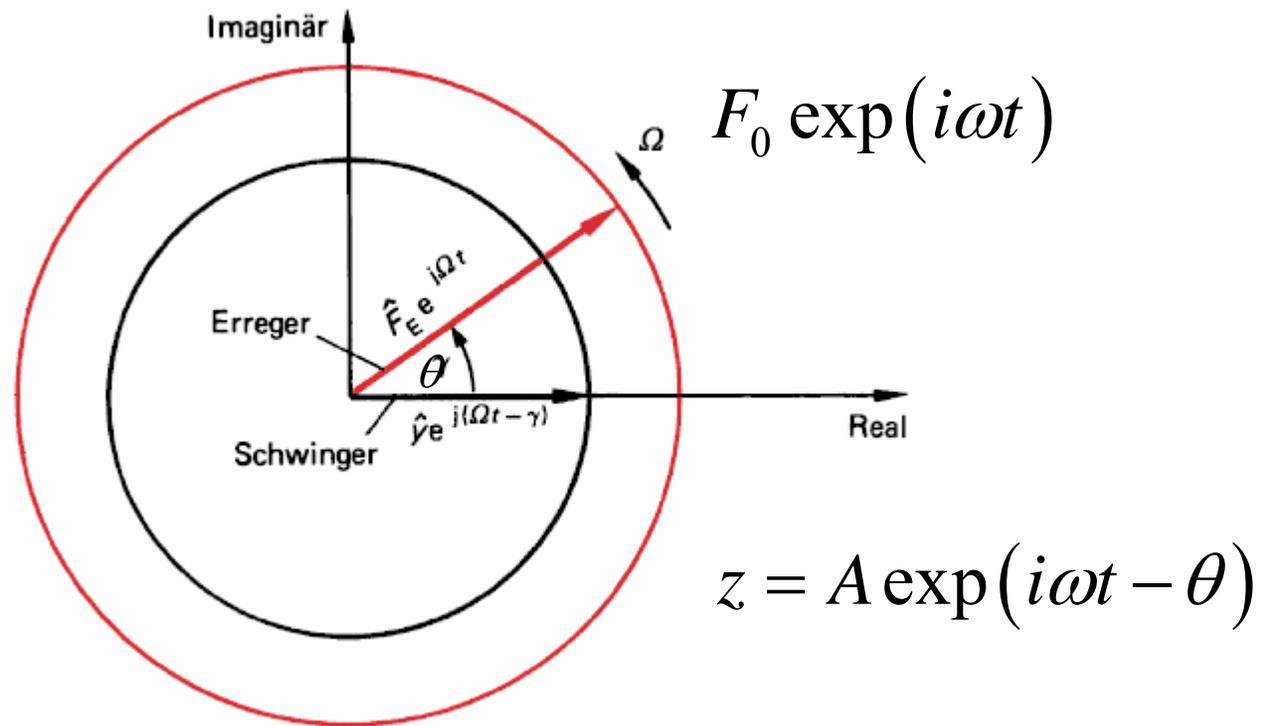
$$m \frac{d^2 x}{dt^2} = -b \frac{dx}{dt} - kx + F_0 \cos(\omega t) \frac{|A|}{F_0}$$

$$m \frac{d^2 z}{dt^2} + b \frac{dz}{dt} + kz = F_0 e^{i\omega t}$$

$$x = \operatorname{Re}(z)$$



$$m \frac{d^2 z}{dt^2} + b \frac{dz}{dt} + kz = F_0 e^{i\omega t}$$



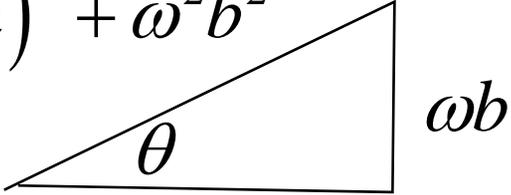
Resonanz

$$m \frac{d^2 z}{dt^2} + b \frac{dz}{dt} + kz = F_0 e^{i\omega t}$$

$$z = A e^{i\omega t}$$

$$-\omega^2 m A e^{i\omega t} + i\omega b A e^{i\omega t} + k A e^{i\omega t} = F_0 e^{i\omega t}$$

$$A(-\omega^2 m + i\omega b + k) = F_0$$

$$\rho = \sqrt{(k - \omega^2 m)^2 + \omega^2 b^2}$$


$k - \omega^2 m$

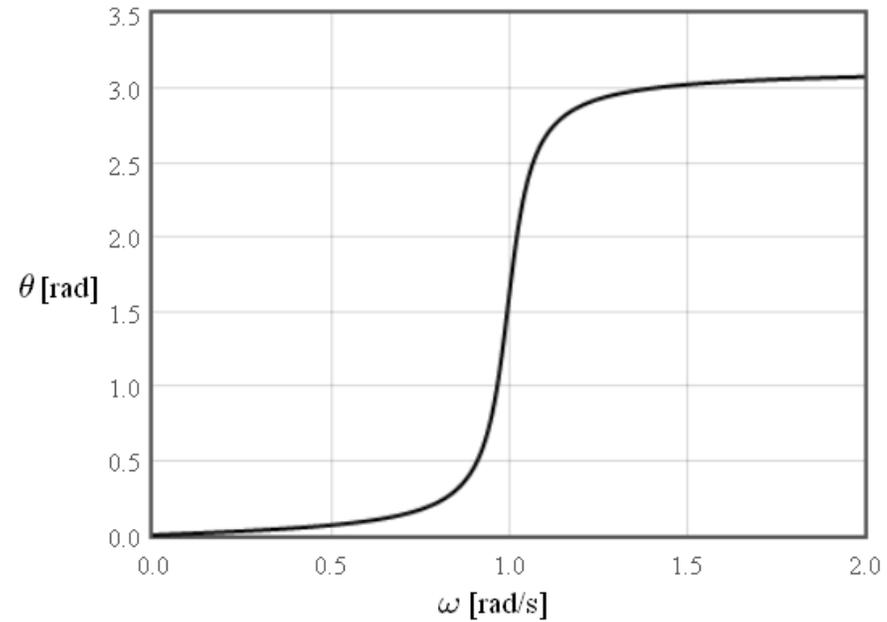
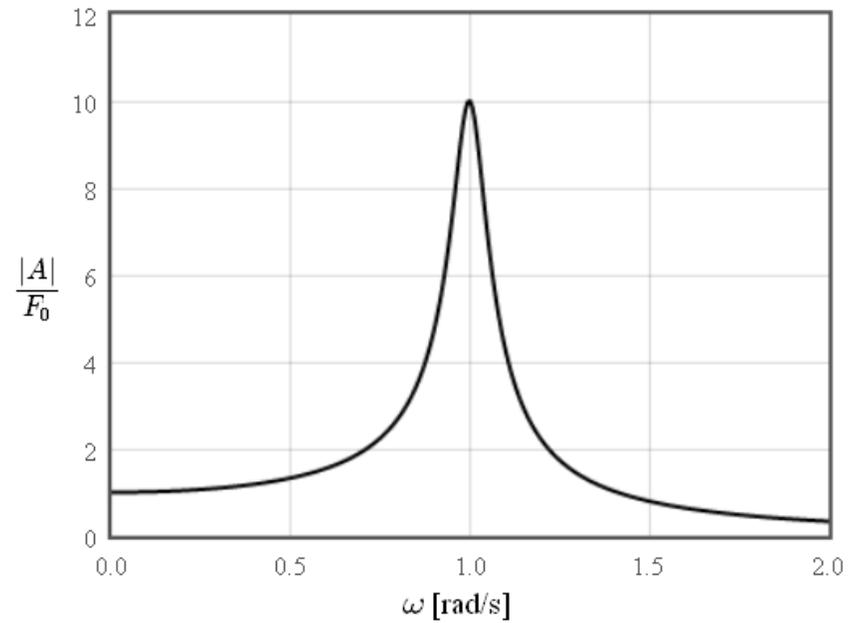
$$|A| = \frac{F_0}{\rho}$$

$m =$ **[kg]** $b =$ **[N s/m]** $k =$ **[N/m]**
 $Q = \frac{\sqrt{mk}}{b} =$

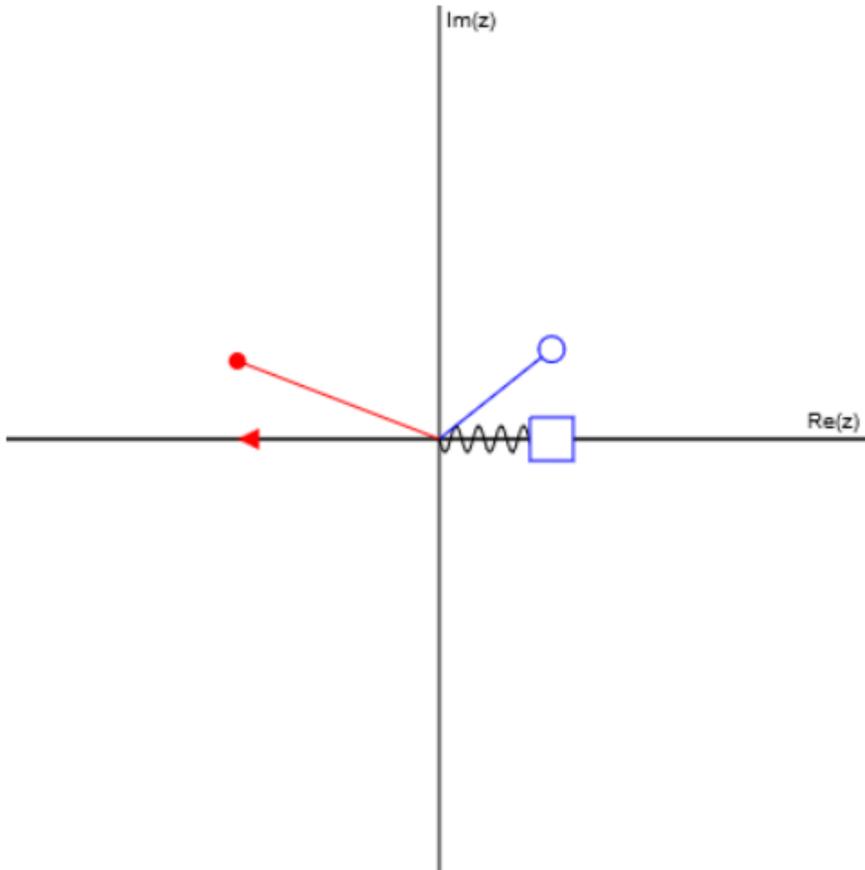
$$\frac{|A|}{F_0} = \frac{1}{\sqrt{(k - m\omega^2)^2 + \omega^2 b^2}}$$

$$A = \frac{F_0}{\rho} e^{-i\theta}$$

$$\tan \theta = \frac{\omega b}{k - m\omega^2}$$



Resonanz



$$m = 4 \text{ [kg]}$$

$$b = 1 \text{ [N s/m]}$$

$$k = 6 \text{ [N/m]}$$

$$F_0 = 1 \text{ [N]}$$

$$\omega = 1.3 \text{ [rad/s]}$$

$$\omega_0 = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = 1.22 \text{ [rad/s]} = 0.194 \text{ [Hz]}$$

$$\theta = \text{atan}\left(\frac{\omega b}{k - m\omega^2}\right) = 2.10 \text{ [rad]} = 120 \text{ [deg]}$$

$$A = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \omega^2 b^2}} = 0.664 \text{ [m]}$$

$$Q = \frac{\sqrt{mk}}{b} = 4.90$$

$F_0 e^{i\omega t}$ anzeigen:

$A e^{i(\omega t - \theta)}$ anzeigen:

z anzeigen:

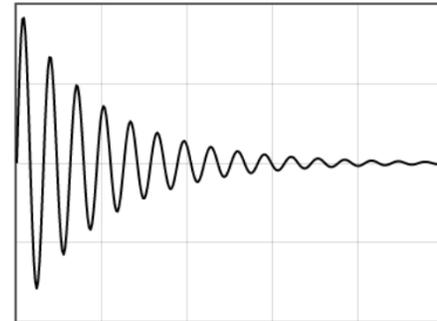
x_2 anzeigen:

Gütefaktor

$$Q = \frac{\pi\tau}{T}$$

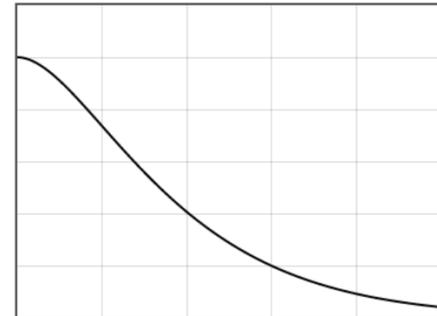
Schwingfall

$$Q > \frac{1}{2}$$



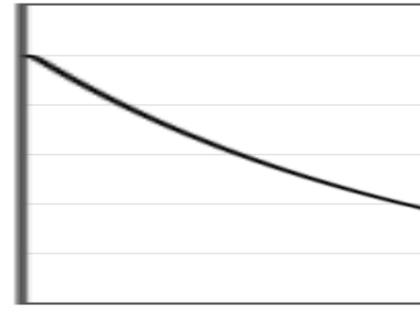
aperiodischer
Grenzfall

$$Q = \frac{1}{2}$$

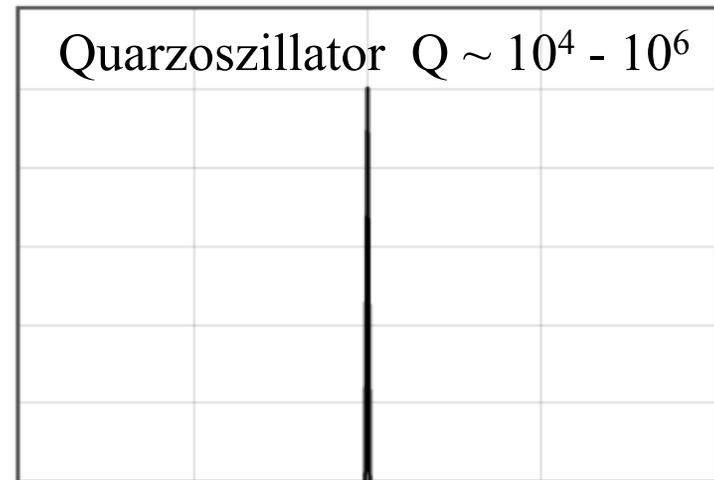
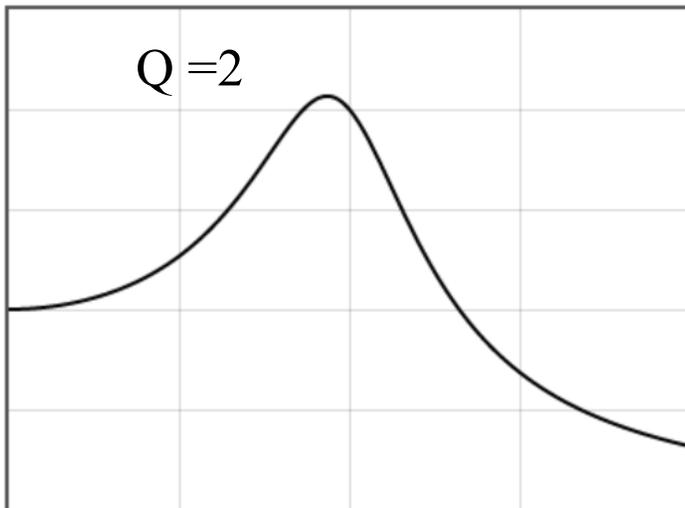
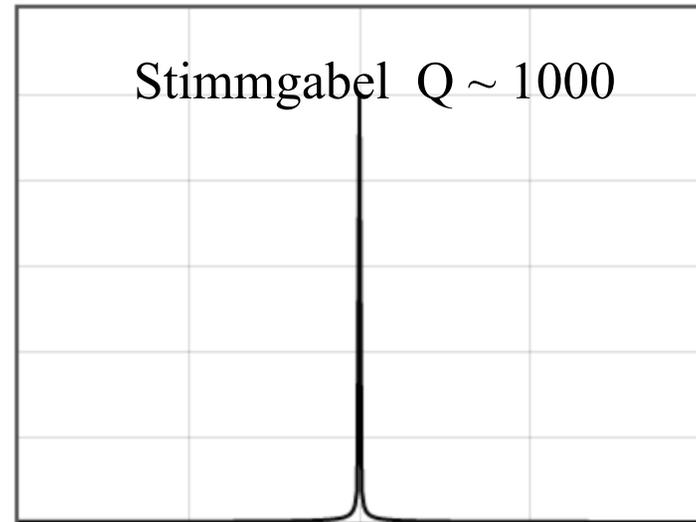
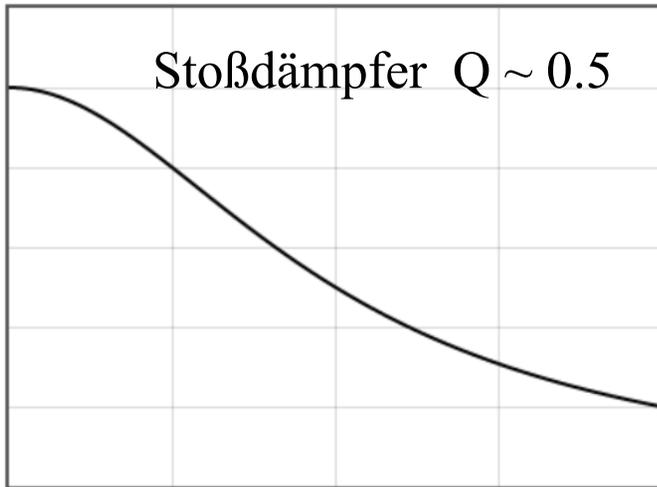


Kriechfall

$$Q < \frac{1}{2}$$

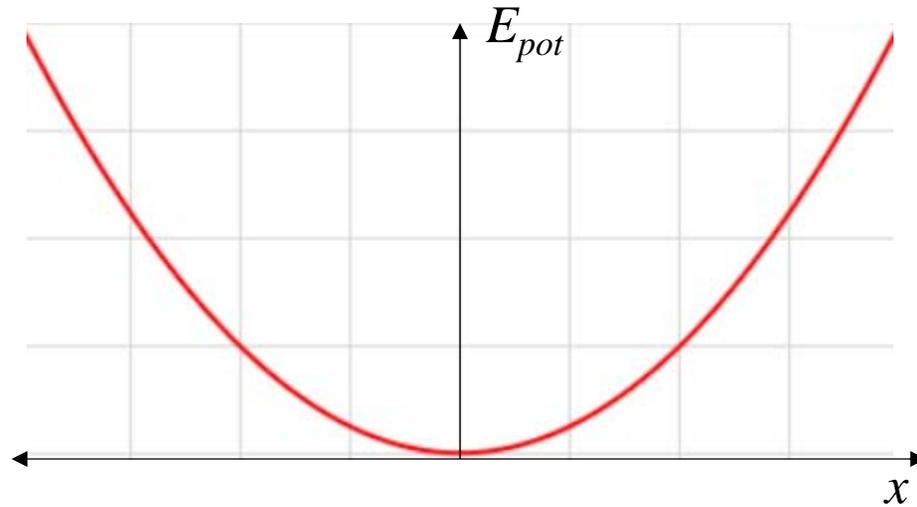


Gütefaktor



andere Schwingungssysteme

E_{pot} hat ein Minimum bei dem Gleichgewichtspunkt



$$F_x = -\frac{dU}{dx}$$

Schwingungssysteme

für kleine Amplitudenschwankungen und niedrigen Geschwindigkeiten die Differentialgleichung für eine schwingende Systeme ist

$$a \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + cx = d$$

Lineare gewöhnliche Differentialgleichung zweiter Ordnung

Lineare gewöhnliche Differentialgleichung

$$m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = 0$$

Lineare Differentialgleichung zweiter Ordnung

$$m \left(\frac{d^2 x}{dt^2} \right)^2 + b \frac{dx}{dt} + kx = 0$$

Nichtlineare
Differentialgleichung
zweiter Ordnung

$$m \frac{d^2 x}{dt^2} + b \left(\frac{dx}{dt} \right)^2 + kx = 0$$

$$m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx^3 = 0$$

Nichtlineare gewöhnliche Differentialgleichung

Numerisches Lösen von Differentialgleichungen 2. Ordnung

$$\frac{dx}{dt} = v_x$$
$$a_x = \frac{F_x}{m} = \frac{dv_x}{dt} = -x*x*x$$

Anfangsbedingungen:

$$x(t_0) = 0$$

$$\Delta t = 0.05$$

$$v_x(t_0) = 1$$

$$N_{steps} = 500$$

$$t_0 = 0$$

Graphische Darstellung: x vs. t

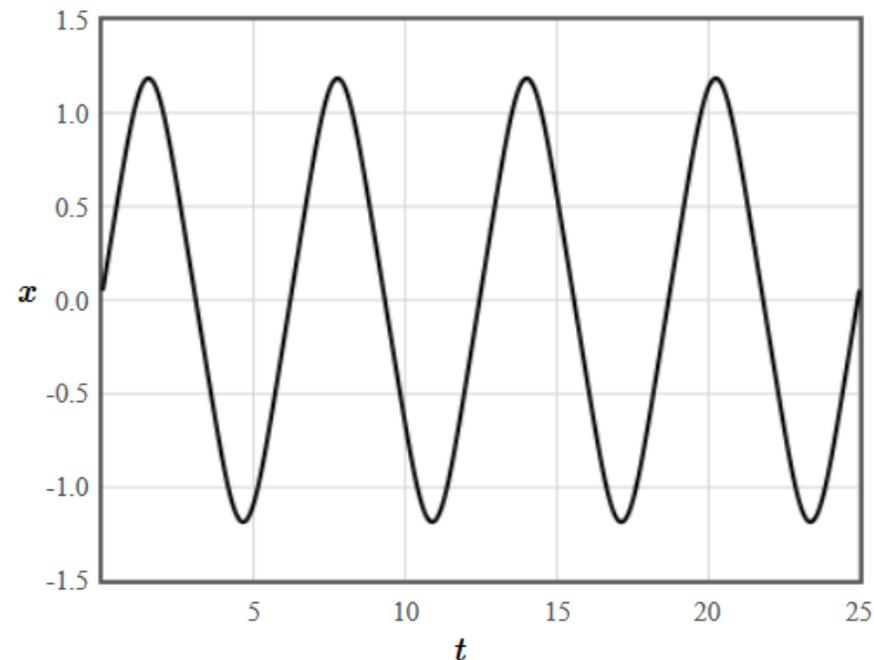
Absenden

harte Feder

$$F_{Feder} = -kx^3$$

weiche Feder

$$F_{Feder} = -k \tanh(x)$$



Nichtlineare gewöhnliche Differentialgleichung

Numerisches Lösen von Differentialgleichungen 2. Ordnung

$$\frac{dx}{dt} = v_x$$
$$a_x = \frac{F_x}{m} = \frac{dv_x}{dt} = -0.3*v_x*v_x*v_x - x*x*x$$

Anfangsbedingungen:

$$x(t_0) = 0$$

$$\Delta t = 0.05$$

$$v_x(t_0) = 1$$

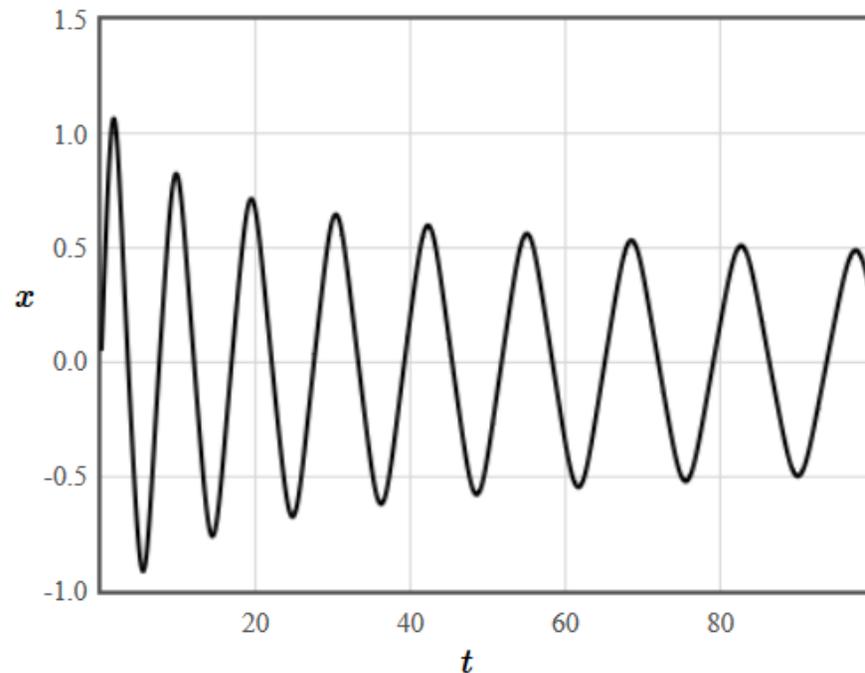
$$N_{steps} = 2000$$

$$t_0 = 0$$

Graphische Darstellung: x vs. t

Absenden

$$m \frac{d^2 x}{dt^2} + b \left(\frac{dx}{dt} \right)^3 + kx^3 = 0$$



Nichtlineare Schwinger

Numerisches Lösen von Differentialgleichungen 2. Ordnung

$$\frac{dx}{dt} = v_x$$
$$a_x = \frac{F_x}{m} = \frac{dv_x}{dt} = -0.1*v_x - \sin(x)$$

Anfangsbedingungen:

$$x(t_0) = 0$$

$$v_x(t_0) = 1$$

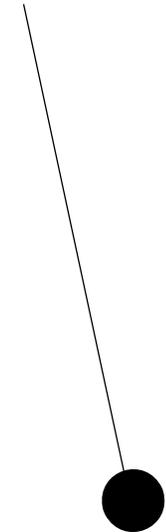
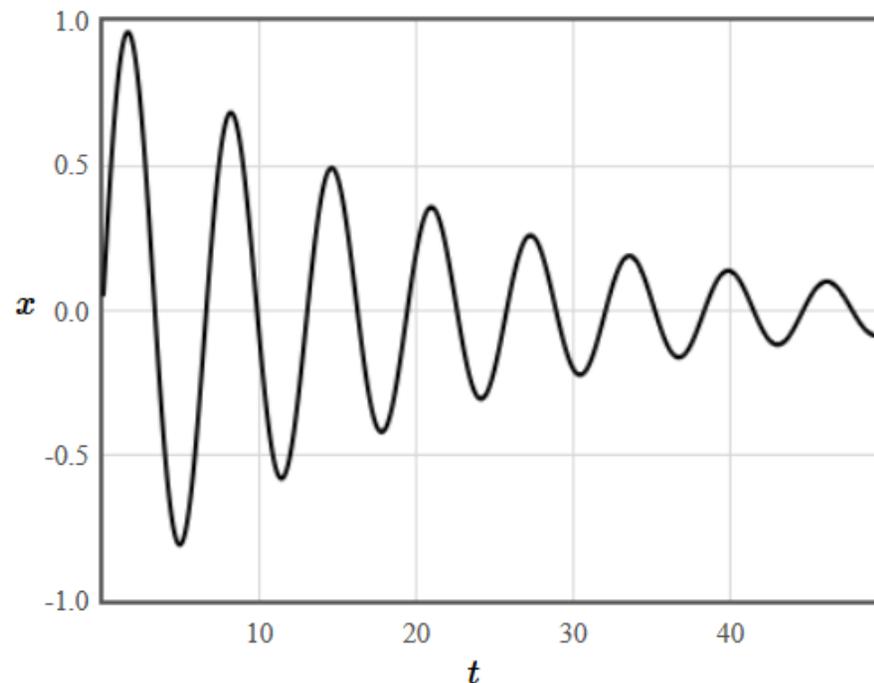
$$t_0 = 0$$

$$\Delta t = 0.05$$

$$N_{steps} = 1000$$

Graphische Darstellung: x vs. t

Absenden



Pendel

weiche Feder

Parametrisch erregte Schwingungen

Numerisches Lösen von Differentialgleichungen 2. Ordnung

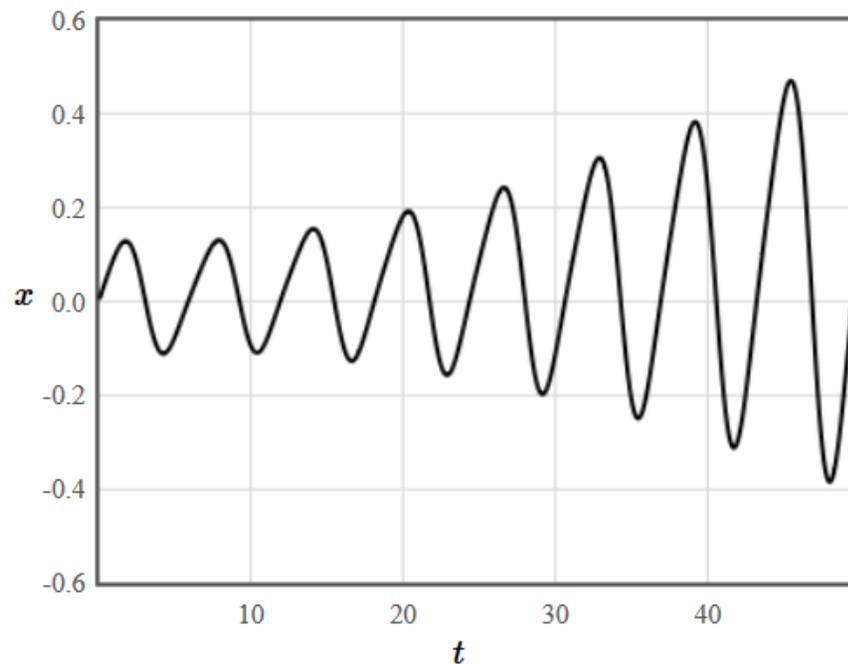
$$\frac{dx}{dt} = v_x$$
$$a_x = \frac{F_x}{m} = \frac{dv_x}{dt} = -0.1*v_x - (1 - \cos(t)) * \sin(x)$$

Anfangsbedingungen:

$x(t_0) = 0$ $\Delta t = 0.05$

$v_x(t_0) = 0.1$ $N_{steps} = 1000$

$t_0 = 0$ Graphische Darstellung: x vs. t



Kind auf einer Schaukel

Nichtlineare Schwinger

Numerisches Lösen von Differentialgleichungen 2. Ordnung

$$\frac{dx}{dt} = v_x$$
$$a_x = \frac{F_x}{m} = \frac{dv_x}{dt} = (1-x^2)*v_x-x$$

Anfangsbedingungen:

$$x(t_0) = 0$$

$$v_x(t_0) = 1$$

$$t_0 = 0$$

$$\Delta t = 0.05$$

$$N_{steps} = 1000$$

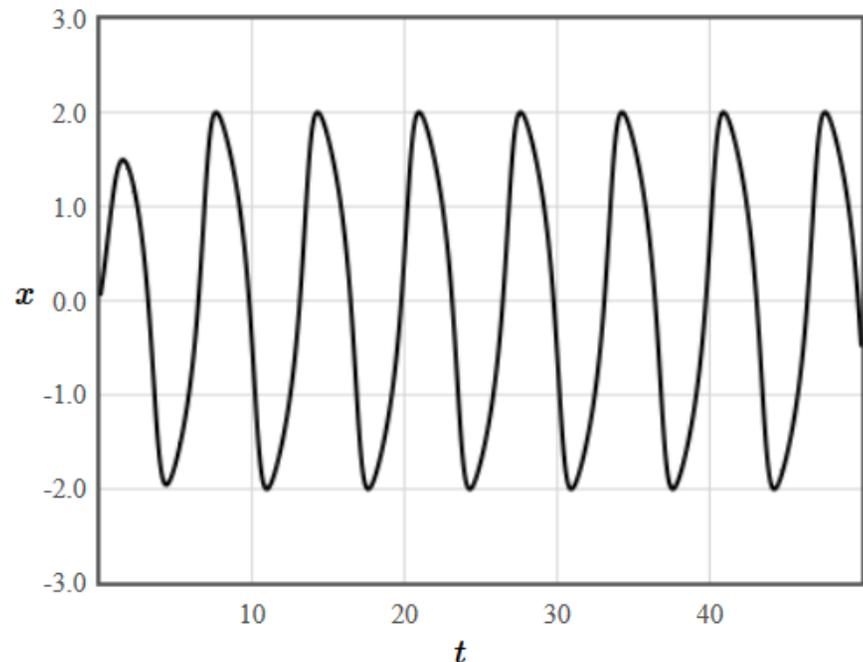
Graphische Darstellung: vs.

Absenden

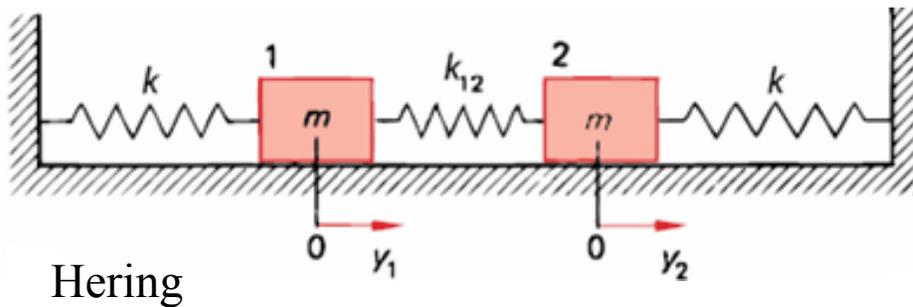
Van-der-Pol-System

$$\frac{d^2 x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = 0$$

negativen Reibung für kleine x



gekoppeltes Schwingungssystem



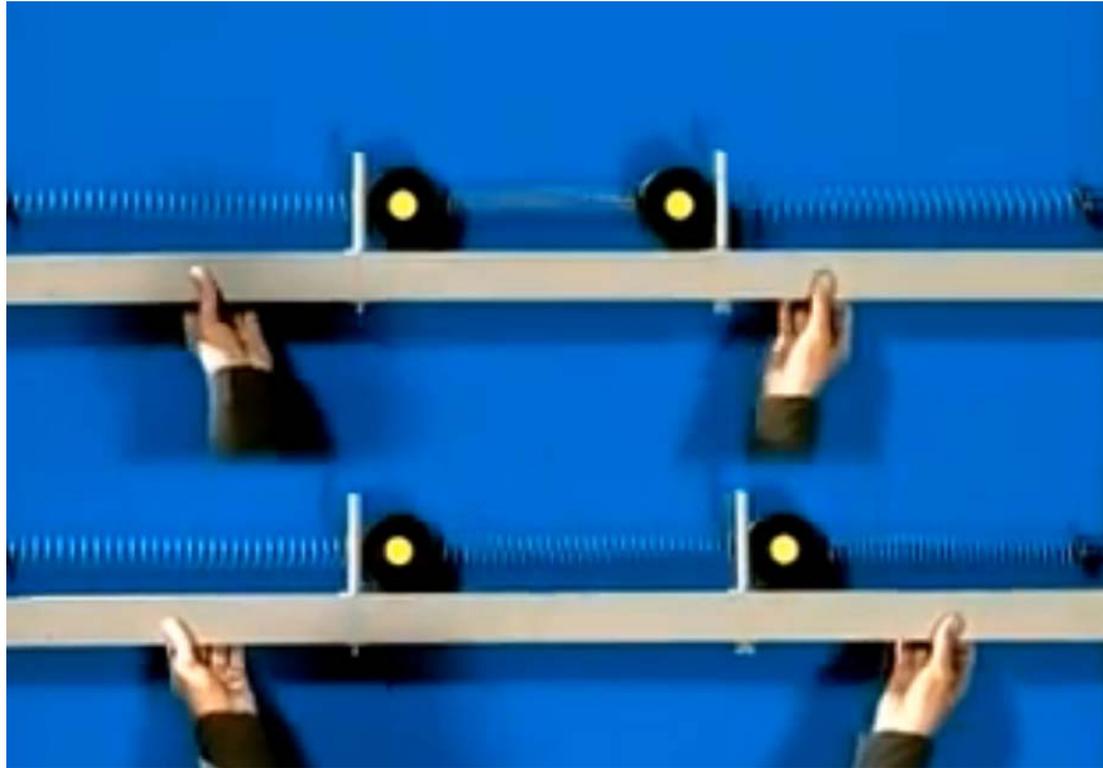
$$M \frac{d^2 y_1}{dt^2} = -ky_1 + k_{12}(y_2 - y_1)$$

$$M \frac{d^2 y_2}{dt^2} = -ky_2 + k_{12}(y_1 - y_2)$$

Eigenmoden: harmonische Bewegung

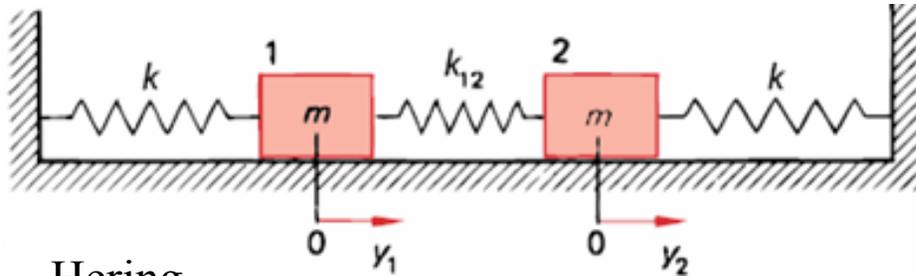
Alle Teile schwingen mit der gleichen Frequenz

zwei Eigenmoden



<https://www.youtube.com/watch?v=8RV3gXm6j2I>

gekoppeltes Schwingungssystem



Hering

$$M \frac{d^2 y_1}{dt^2} = -ky_1 + k_{12}(y_2 - y_1)$$

$$M \frac{d^2 y_2}{dt^2} = -ky_2 + k_{12}(y_1 - y_2)$$

Numerisches Lösen von Differentialgleichungen 6. Ordnung

$$\frac{dx}{dt} = v_x$$

$$\frac{dv_x}{dt} = -x - 0.3*(y-x)$$

$$\frac{dy}{dt} = v_y$$

$$\frac{dv_y}{dt} = -y - 0.3*(x-y)$$

$$\frac{dz}{dt} = v_z$$

$$\frac{dv_z}{dt} = 0$$

Anfangsbedingungen:

$$t_0 = 0$$

$$x(t_0) = 0$$

$$v_x(t_0) = 1$$

$$y(t_0) = 0$$

$$v_y(t_0) = 1$$

$$z(t_0) = 0$$

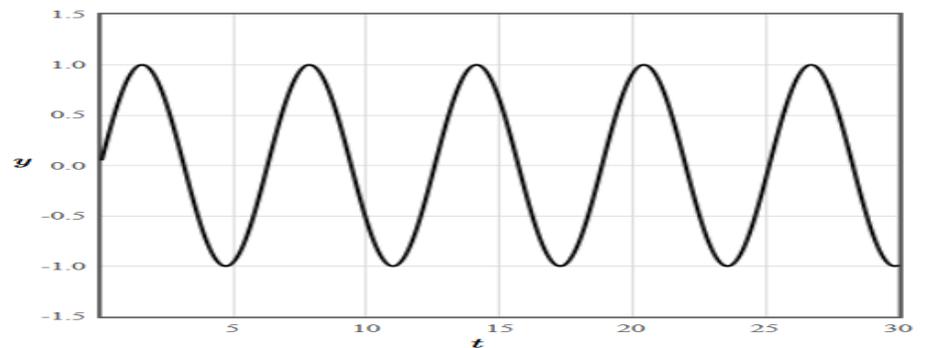
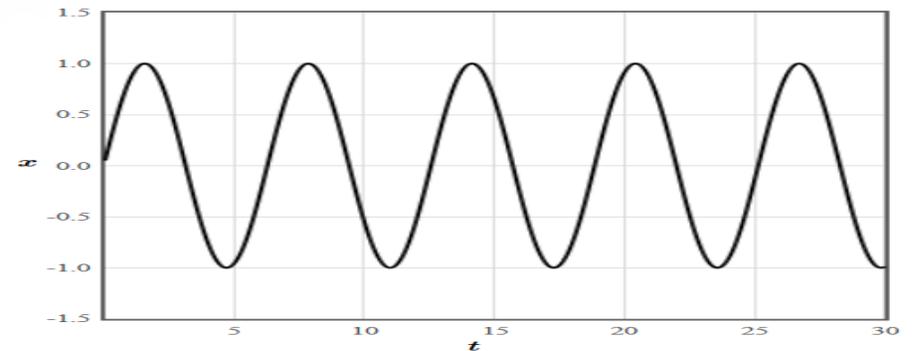
$$v_z(t_0) = 0$$

$$\Delta t = 0.05$$

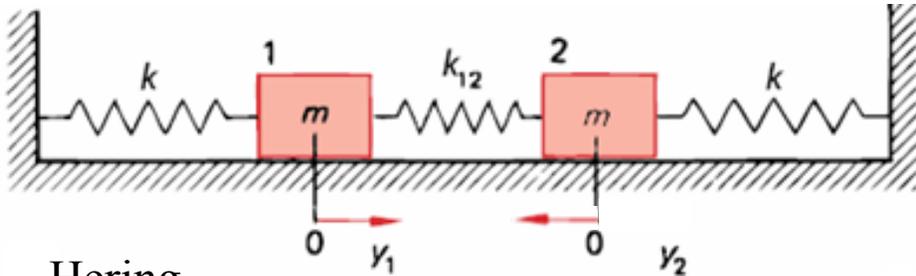
$$N_{steps} = 600$$

Graphische Darstellung: x vs. t

Absenden



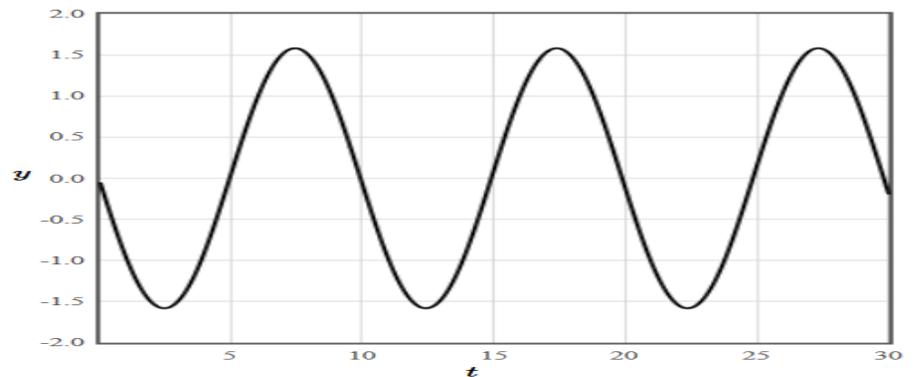
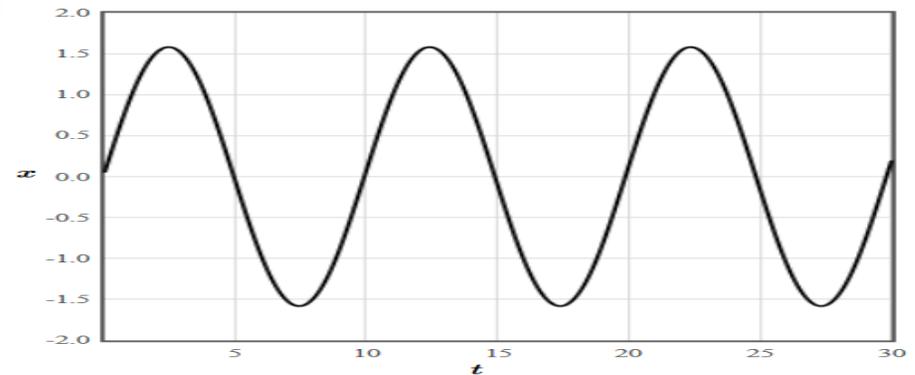
gekoppeltes Schwingungssystem



Hering

$$M \frac{d^2 y_1}{dt^2} = -ky_1 + k_{12}(y_2 - y_1)$$

$$M \frac{d^2 y_2}{dt^2} = -ky_2 + k_{12}(y_1 - y_2)$$



Numerisches Lösen von Differentialgleichungen 6. Ordnung

$$\frac{dx}{dt} = v_x$$

$$\frac{dv_x}{dt} = -x - 0.3 \cdot (y - x)$$

$$\frac{dy}{dt} = v_y$$

$$\frac{dv_y}{dt} = -y - 0.3 \cdot (x - y)$$

$$\frac{dz}{dt} = v_z$$

$$\frac{dv_z}{dt} = 0$$

Anfangsbedingungen:

$$t_0 = 0$$

$$\Delta t = 0.05$$

$$x(t_0) = 0$$

$$N_{steps} = 600$$

$$v_x(t_0) = 1$$

Graphische Darstellung: vs.

$$y(t_0) = 0$$

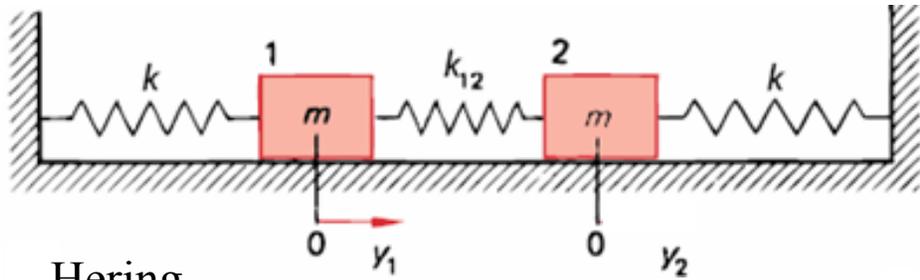
$$v_y(t_0) = -1$$

$$z(t_0) = 0$$

$$v_z(t_0) = 0$$

Absenden

gekoppeltes Schwingungssystem



Hering

$$M \frac{d^2 y_1}{dt^2} = -ky_1 + k_{12}(y_2 - y_1)$$

$$M \frac{d^2 y_2}{dt^2} = -ky_2 + k_{12}(y_1 - y_2)$$

Numerisches Lösen von Differentialgleichungen 6. Ordnung

$$\frac{dx}{dt} = v_x$$

$$\frac{dv_x}{dt} = (-x + 0.1*(y-x))$$

$$\frac{dy}{dt} = v_y$$

$$\frac{dv_y}{dt} = (-y + 0.1*(x-y))$$

$$\frac{dz}{dt} = v_z$$

$$\frac{dv_z}{dt} = 0$$

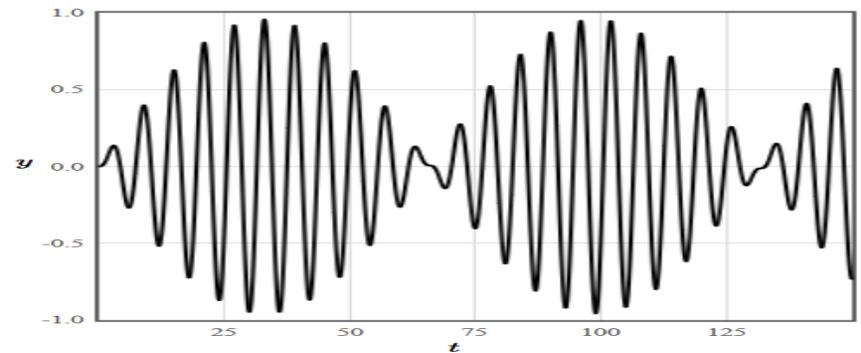
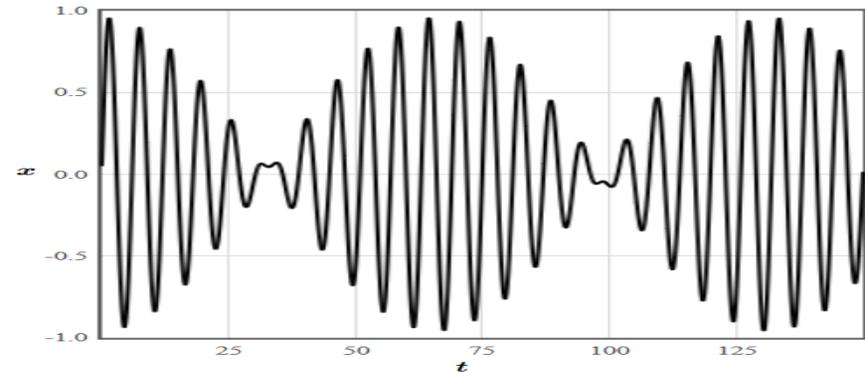
Anfangsbedingungen:

$t_0 = 0$
 $x(t_0) = 0$
 $v_x(t_0) = 1$
 $y(t_0) = 0$
 $v_y(t_0) = 0$
 $z(t_0) = 0$
 $v_z(t_0) = 0$

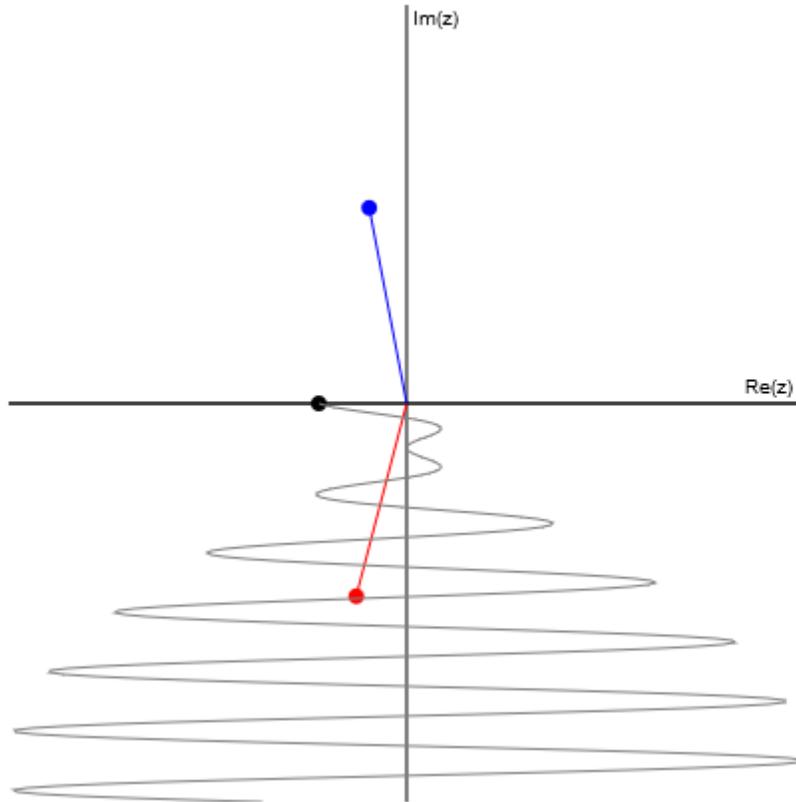
$\Delta t = 0.05$
 $N_{steps} = 3000$

Graphische Darstellung: vs.

Absenden



Schwebung



$$x = A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t).$$

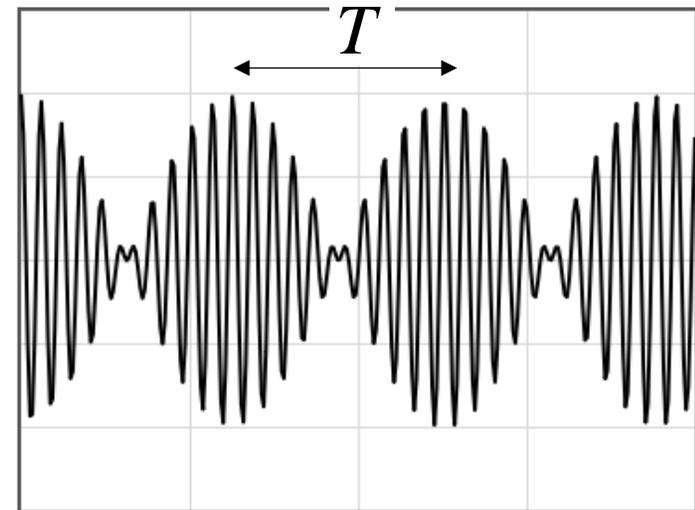
$$A_1 = 1 \text{ [m]}$$

$$A_2 = 1 \text{ [m]}$$

$$\omega_1 = 1 \text{ [rad/s]}$$

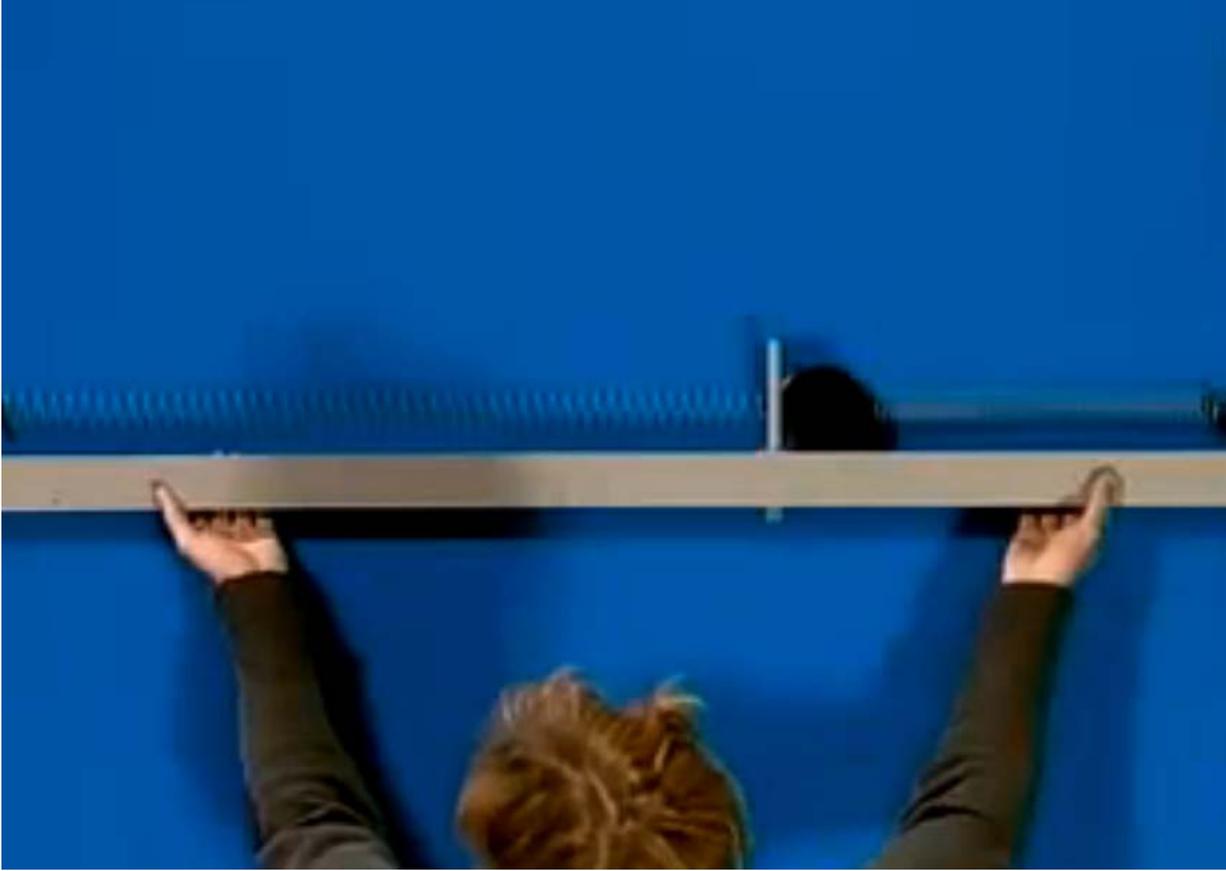
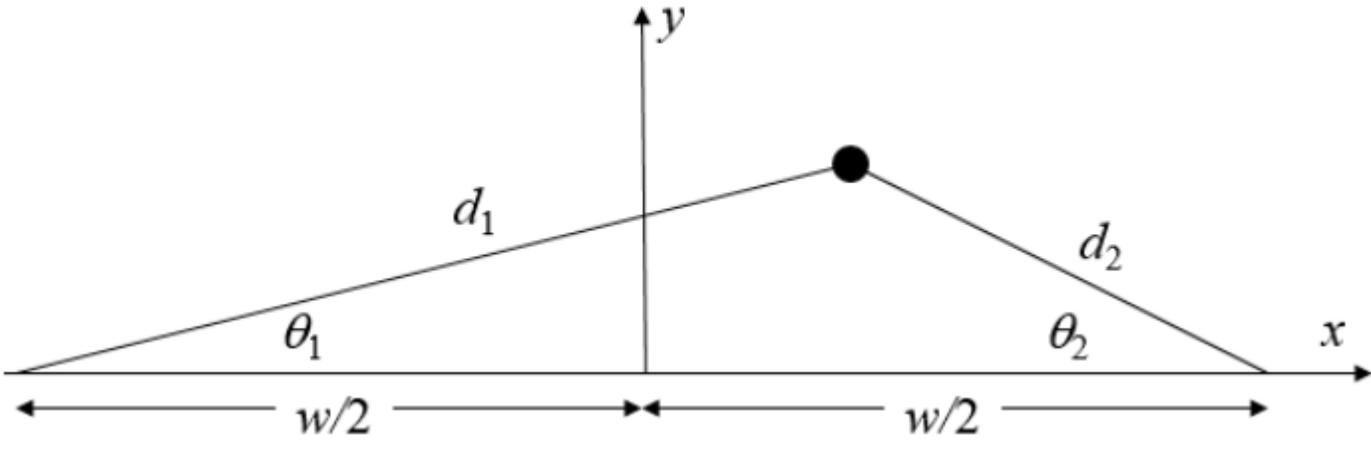
$$\omega_2 = 1.1 \text{ [rad/s]}$$

t = 0



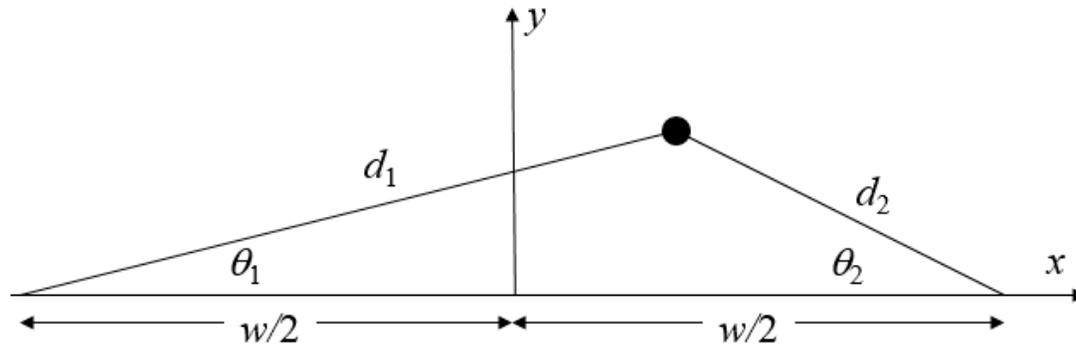
$$\omega_1 T - \omega_2 T = 2\pi$$

$$T = \frac{2\pi}{\omega_1 - \omega_2}$$



A mass on a frictionless table connected to two springs

A mass m on a frictionless horizontal table is connected to the left and to the right by two springs each with a spring constant k . The length of the unstretched springs is l and the width of the table is $w > 2l$.



The equilibrium position for the mass is in the middle of the table ($x = 0, y = 0$) where the forces exerted by the springs cancel each other out. If the mass is moved away from its equilibrium position then the force on the mass is,

$$\vec{F} = k[(d_2 - l) \cos \theta_2 - (d_1 - l) \cos \theta_1] \hat{x} - k[(d_2 - l) \sin \theta_2 + (d_1 - l) \sin \theta_1] \hat{y}.$$

Here $d_1 = \sqrt{(w/2 + x)^2 + y^2}$, $d_2 = \sqrt{(w/2 - x)^2 + y^2}$, $\theta_1 = \text{atan}\left(\frac{y}{w/2+x}\right)$, and $\theta_2 = \text{atan}\left(\frac{y}{w/2-x}\right)$.

The form below will calculate the trajectory of the mass when it is released from position (x_0, y_0) at rest.

$m =$ $\text{kg} \quad k =$ $\text{N/m} \quad l =$ $\text{m} \quad w =$ $\text{m} \quad x_0 =$ $\text{m} \quad y_0 =$ m

Load these equations into the form below

Numerical 6th order differential equation solver

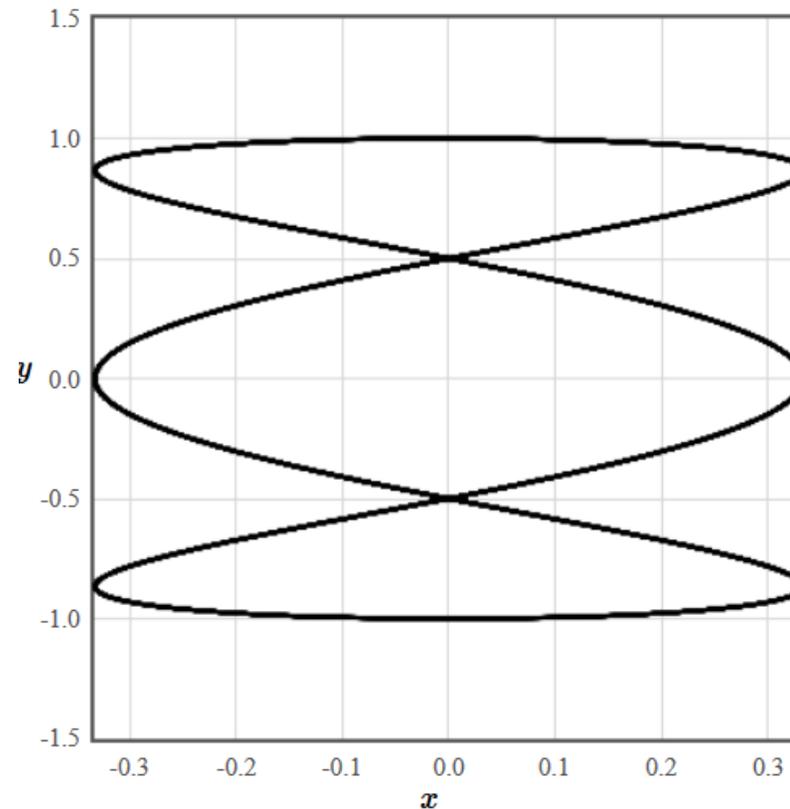
$$\frac{dx}{dt} = v_x$$

$$\frac{dv_x}{dt} = \frac{(0.2 * (\text{sqrt}((0.4-x) * (0.4-x) + y * y) - 0.15) * \cos(\text{atan}(y / (0.4-x))) - 0.2 * (\text{sqrt}((0.4+x) * (0.4+x) + y * y) - 0.15) * \cos(\text{atan}(y / (0.4+x))))}{(0.4)}$$

Überlagerung harmonischer Schwingungen mit ganzzahligem Frequenzverhältnis, die senkrecht zueinander schwingen (Lissajous-Figuren)

$$M \frac{d^2 x}{dt^2} = -k_2 x$$

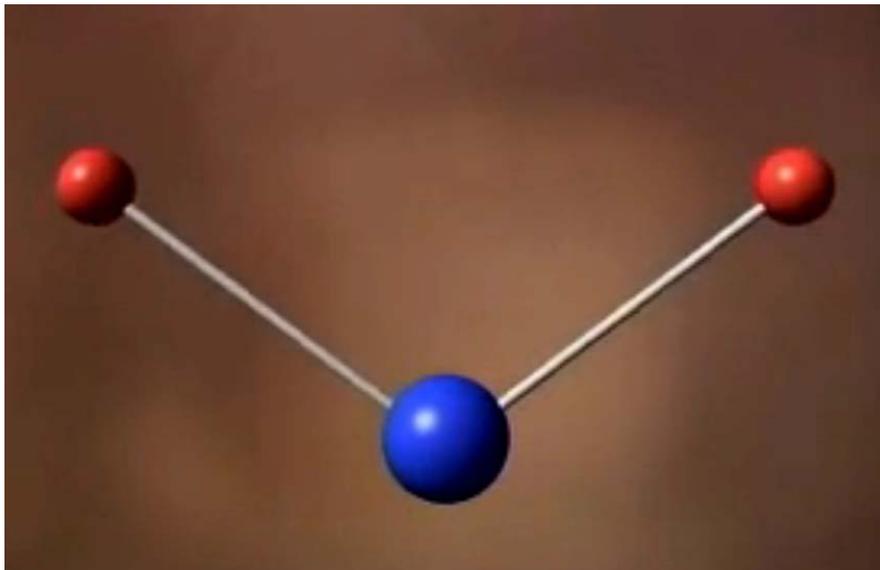
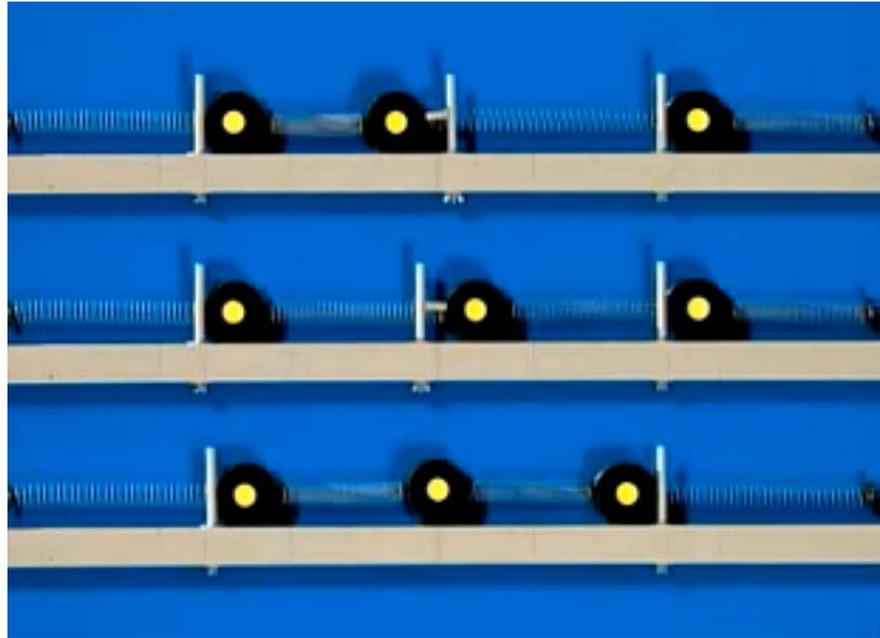
$$M \frac{d^2 y}{dt^2} = -k_2 y$$



3 Eigenmoden

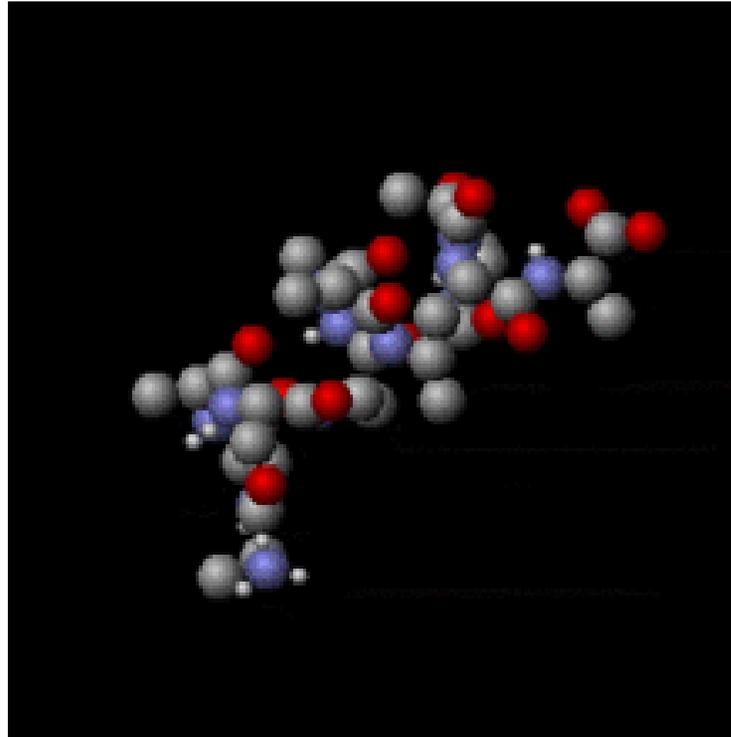
jede Bewegung kann als eine Superposition von Eigenmoden geschrieben werden.

Eigenmoden:
harmonische Bewegung
Alle Teile schwingen mit der gleichen Frequenz



Eigenmoden

jede Bewegung kann als eine Superposition von Eigenmoden geschrieben werden.



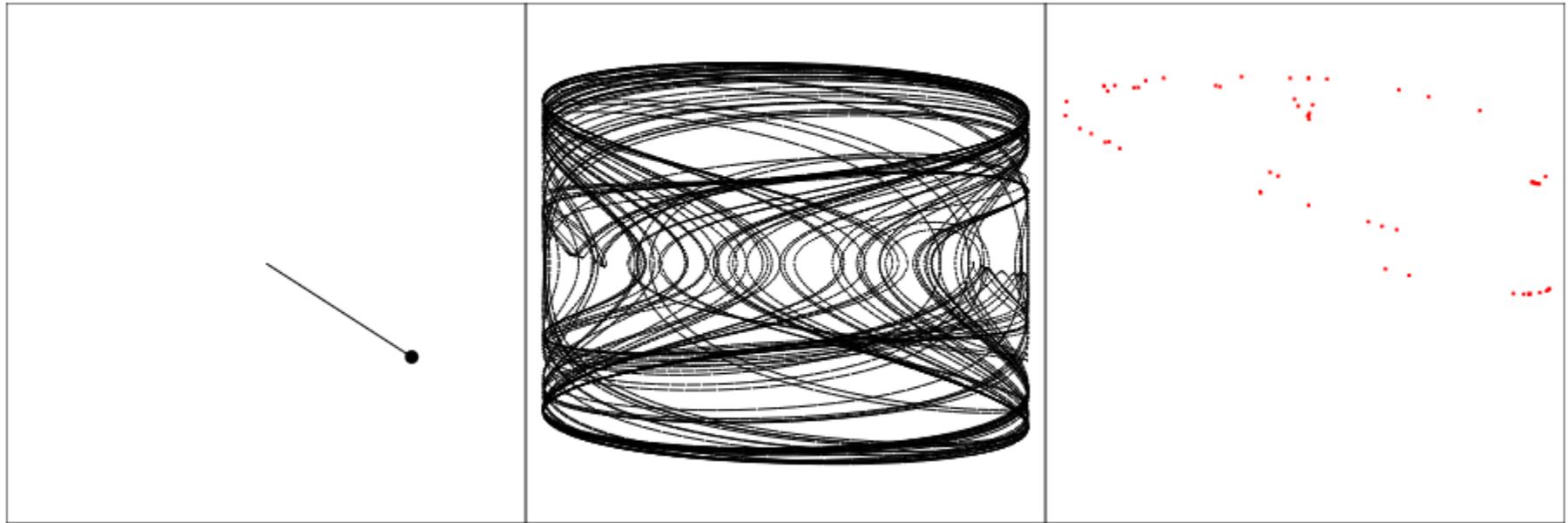
3 Translationsfreiheitsgrade

3 Rotationsfreiheitsgrade

$3N - 6$ Eigenmoden
(Schwingunen)

The driven pendulum

$$\frac{d^2\theta}{dt^2} + \frac{1}{q} \frac{d\theta}{dt} + \sin(\theta) = A \cos(\omega t)$$

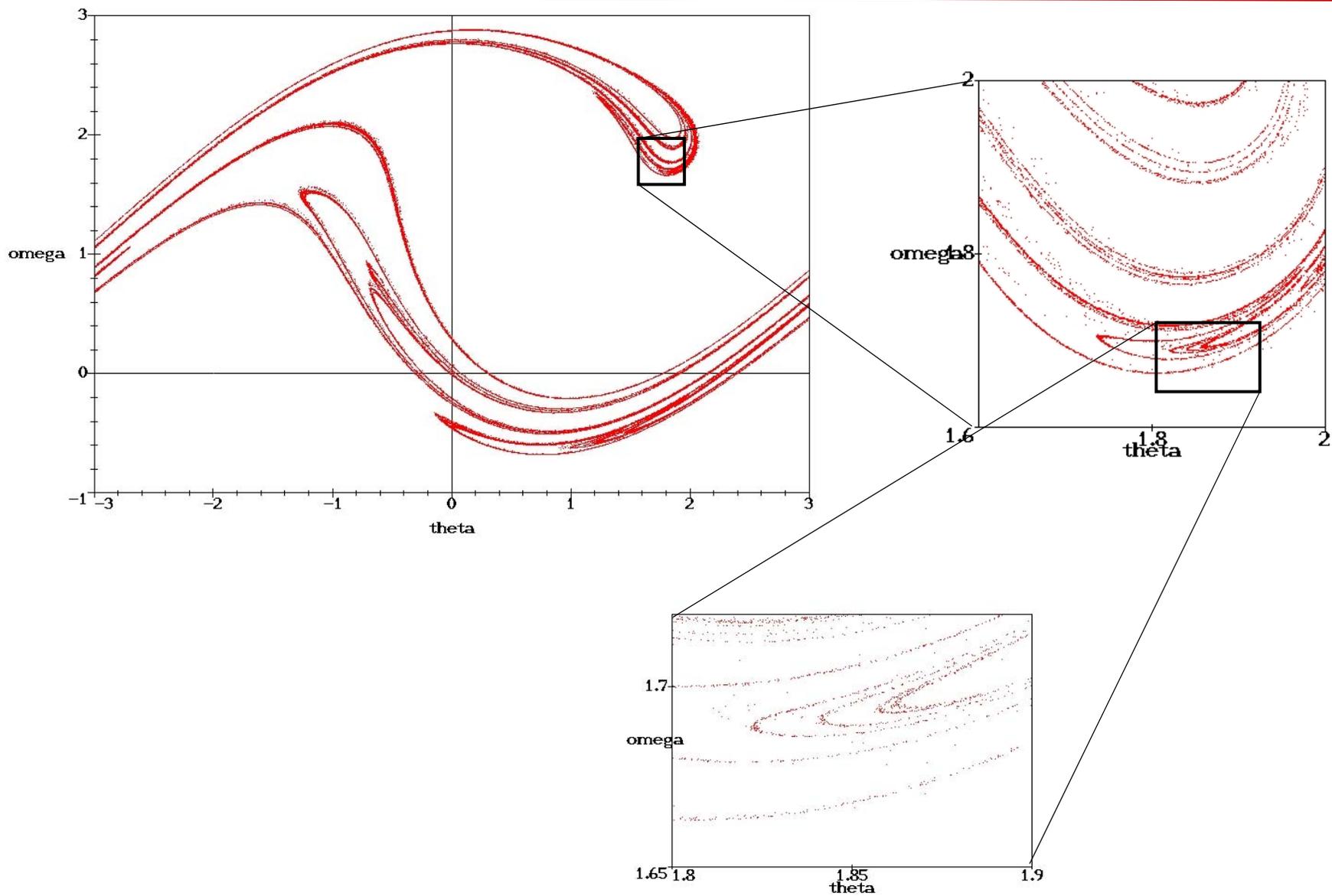


$q = 2$
 $A = 1.15$
 $\omega = 0.67$

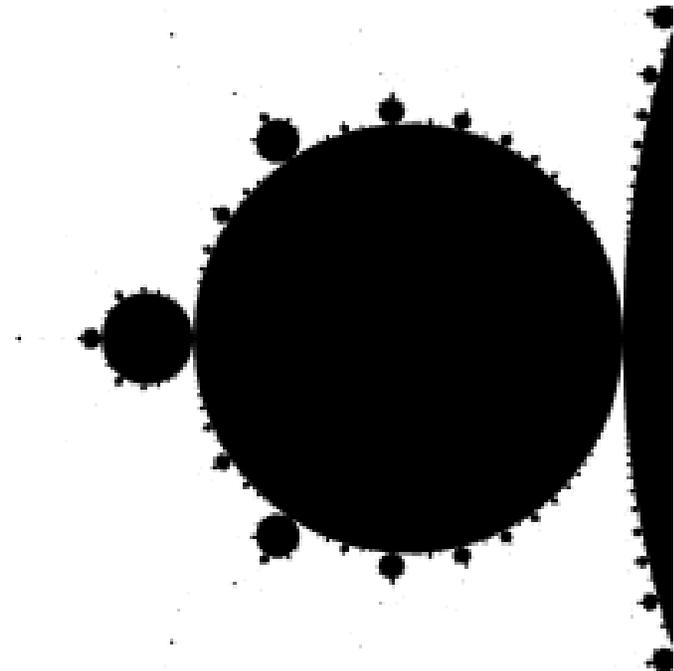
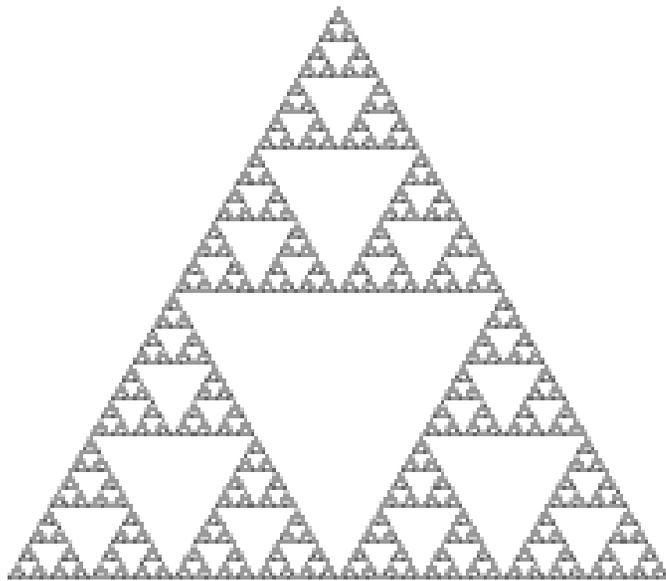
Poincaré map

Symmetriebrechung \rightarrow $A = 0, 0.6, 1.04, 1.08, 1.1, 1.15$ \leftarrow Chaos
Periodenverdopplung \uparrow

Seltsamer Attraktor



Selbstähnlichkeit



<http://en.wikipedia.org/wiki/Self-similarity>