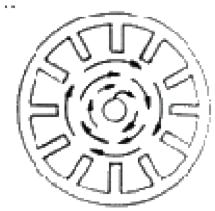
Charged particle in a magnetic field

$$\vec{F} = -e\vec{v} \times \vec{B}$$

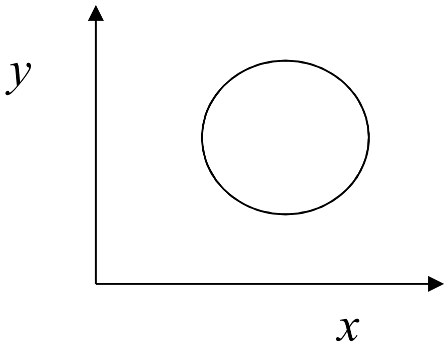
$$evB_z = \frac{mv^2}{R}$$

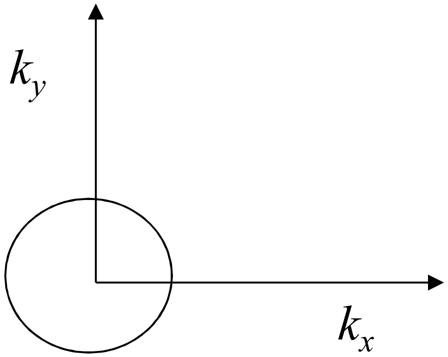
$$v = \omega_c R$$

$$\omega_c = \frac{eB_z}{m}$$



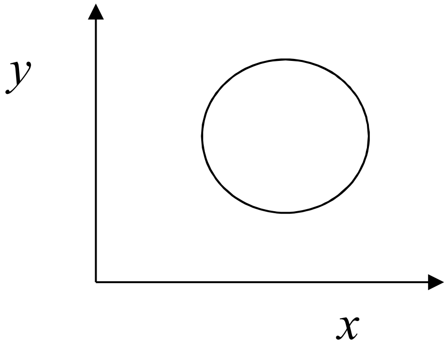
Magnetron



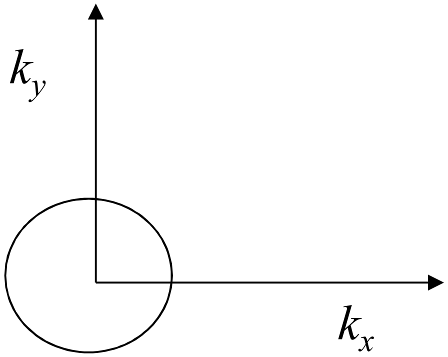


Why don't metals in a B field radiate?

Magnetron



$$\omega_c = \frac{eB_z}{m}$$



There are no lower lying states to fall into. We need a quantum description.

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

Magnetic force depends on the velocity, not on the position.

Electrons in a magnetic field

Lorentz force law $\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$

Euler Lagrange equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$

Lagrangian: $L = \tfrac{1}{2} m v^2 - qV\left(\vec{r}, t\right) + q\vec{v} \cdot \vec{A}\left(\vec{r}, t\right)$

Kittel: Appendix G
http://lamp.tu-graz.ac.at/~hadley/ss2/IQHE/cpimf.php

Lagrangian

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)-\frac{\partial L}{\partial x_i}=0 \qquad L=\tfrac{1}{2}mv^2-qV\left(\vec{r},t\right)+q\vec{v}\cdot\vec{A}\left(\vec{r},t\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_x}\right)=\frac{d}{dt}\left(mv_x+qA_x\right)=m\frac{dv_x}{dt}+q\frac{dA_x}{dt}$$

$$=m\frac{dv_x}{dt}+q\left(\frac{dx}{dt}\frac{\partial A_x}{\partial x}+\frac{dy}{dt}\frac{\partial A_x}{\partial y}+\frac{dz}{dt}\frac{\partial A_x}{\partial z}+\frac{\partial A_x}{\partial t}\right)$$

$$=m\frac{dv_x}{dt}+q\left(v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_x}{\partial y}+v_z\frac{\partial A_x}{\partial z}+\frac{\partial A_x}{\partial t}\right),$$

$$\frac{\partial L}{\partial x}=-q\frac{\partial V}{\partial x}+q\left(v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_y}{\partial x}+v_z\frac{\partial A_z}{\partial x}\right)$$

$$m\frac{dv_x}{dt}=-q\left(\frac{\partial V}{\partial x}+\frac{\partial A_x}{\partial t}\right)+q\left(v_y\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)+v_z\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right)\right)$$

Lagrangian

$$m\frac{dv_x}{dt} = -q\left(\frac{\partial V}{\partial x} + \frac{\partial A_x}{\partial t}\right) + q\left(v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + v_z\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\right)$$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \qquad \text{and} \qquad \vec{B} = \nabla \times \vec{A}$$

$$m\frac{dv_x}{dt} = q\left(E_x + \left(\vec{v} \times \vec{B}\right)_x\right)$$

$$L = \tfrac{1}{2}mv^2 - qV\left(\vec{r},t\right) + q\vec{v}\cdot\vec{A}\left(\vec{r},t\right)$$

conjugate variable: $$p_x = \frac{\partial L}{\partial v_x} = mv_x + qA_x$$

kinetic momentum

field momentum
(inductance)

$$v_x = \frac{1}{m}\left(p_x - qA_x\right)$$

Hamiltonian

$$\vec{v} = \frac{1}{m}\left(\vec{p} - q\vec{A}\right) \qquad \vec{p} = m\vec{v} + q\vec{A}$$

$$L = \tfrac{1}{2}mv^2 - qV\left(\vec{r},t\right) + q\vec{v}\cdot\vec{A}\left(\vec{r},t\right)$$

Legendre transformation

$$H = \vec{v}\cdot\vec{p} - L$$

$$H = \vec{v}\cdot\left(m\vec{v} + q\vec{A}\right) - \tfrac{1}{2}mv^2 + qV\left(\vec{r},t\right) - q\vec{v}\cdot\vec{A}\left(\vec{r},t\right)$$

$$H = \tfrac{1}{2}mv^2 + qV\left(\vec{r},t\right)$$

$$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + qV\left(\vec{r},t\right)$$

Hamiltonian

$$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + qV$$

Classical result

$$\vec{p} \rightarrow -i\hbar\nabla$$

Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(-i\hbar\nabla - q\vec{A}\right)^2\psi + qV\psi$$

Landau Levels

free particles in a magnetic field

$$\frac{1}{2m}\left(-i\hbar\nabla - q\vec{A}\right)^2 \psi(\vec{r}) = E\psi(\vec{r}). \qquad V=0$$

Landau gauge $\vec{A} = B_z x\hat{y}.$

$$\vec{B} = \nabla \times \vec{A} = B_z x\hat{y} = \left(\frac{dA_z}{dy} - \frac{dA_y}{dz}\right)\hat{x} + \left(\frac{dA_x}{dz} - \frac{dA_z}{dx}\right)\hat{y} + \left(\frac{dA_y}{dx} - \frac{dA_x}{dy}\right)\hat{z}.$$

$$\vec{B} = B_z\hat{z}.$$

Landau Levels

free particles in a magnetic field

$$\frac{1}{2m}\left(-i\hbar\nabla - q\vec{A}\right)^2\psi(\vec{r}) = E\psi(\vec{r}). \qquad V = 0$$

Landau gauge $\vec{A} = B_z x\hat{y}.$

$$\left(-i\hbar\nabla - q\vec{A}\right)^2 = \left(-i\hbar\nabla - qB_z x\hat{y}\right)\cdot\left(-i\hbar\nabla - qB_z x\hat{y}\right)$$

$$-i\hbar\nabla\cdot\left(-qB_z x\hat{y}\right) = -i\hbar\left(\frac{d}{dx}\hat{x} + \frac{d}{dy}\hat{y} + \frac{d}{dz}\hat{z}\right)\cdot\left(-qB_z x\hat{y}\right) = i\hbar qB_z x\frac{d}{dy}$$

$$\frac{1}{2m}\left(-\hbar^2\nabla^2 + i2\hbar qB_z x\frac{d}{dy} + q^2B_z^2x^2\right)\psi = E\psi.$$

The solution has the form

$$\psi = e^{ik_y y}e^{ik_z z}\phi(x).$$

Landau Levels

$$\frac{1}{2m}\left(-\hbar^2\nabla^2 + i2\hbar qB_z x\frac{d}{dy} + q^2B_z^2x^2\right)\psi = E\psi.$$

The solution has the form

$$\psi = e^{ik_y y}e^{ik_z z}\phi(x).$$

Substitute this into the equation

$$\frac{1}{2m}\left(-\hbar^2\frac{\partial^2}{\partial x^2} + \hbar^2k_z^2 + \hbar^2k_y^2 - 2\hbar qB_z k_y x + q^2B_z^2x^2\right)\phi(x) = E\phi(x).$$

Landau Levels

The equation for $\phi(x)$ is

$$\frac{1}{2m}\left(-\hbar^2\frac{\partial^2}{\partial x^2}+\hbar^2k_z^2+\underbrace{\hbar^2k_y^2-2\hbar qB_zk_yx+q^2B_z^2x^2}_{\left(\hbar k_y-qB_zx\right)^2}\right)\phi(x)=E\phi(x).$$

$$\frac{1}{2m}\left(-\hbar^2\frac{\partial^2}{\partial x^2}+q^2B_z^2\left(x-\frac{\hbar k_y}{qB_z}\right)^2\right)\phi(x)=\left(E-\frac{\hbar^2k_z^2}{2m}\right)\phi(x).$$

This is the equation for a harmonic oscillator

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{K}{2}\left(x-x_0\right)^2\right)\phi(x)=E'\phi(x).$$

Landau Levels

$$\frac{1}{2m}\left(-\hbar^2\frac{\partial^2}{\partial x^2}+q^2B_z^2\left(x-x_0\right)^2\right)\phi(x)=\left(E-\frac{\hbar^2k_z^2}{2m}\right)\phi(x).$$

$$x_0=\frac{\hbar k_y}{qB_z}$$

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{K}{2}\left(x-x_0\right)^2\right)\phi(x)=E'\phi(x).$$

This is the equation for a harmonic oscillator

$$\frac{K}{2}\Leftrightarrow\frac{q^2B_z^2}{2m}$$

$$\omega_c=\sqrt{\frac{K}{m}}\Leftrightarrow\frac{qB_z}{m}$$

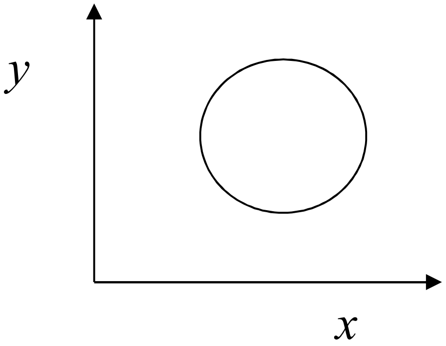
$$E_{k_z,v}=\frac{\hbar^2k_z^2}{2m}+\hbar\omega_c\left(v+\tfrac{1}{2}\right)$$

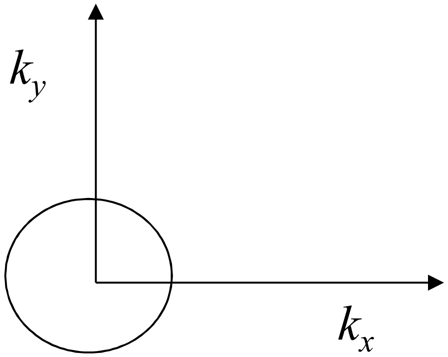
$$\omega_c=\frac{qB_z}{m}$$

Charged particle in a magnetic field

Bohr - Sommerfeld quantization

$$2\pi R = v\lambda$$



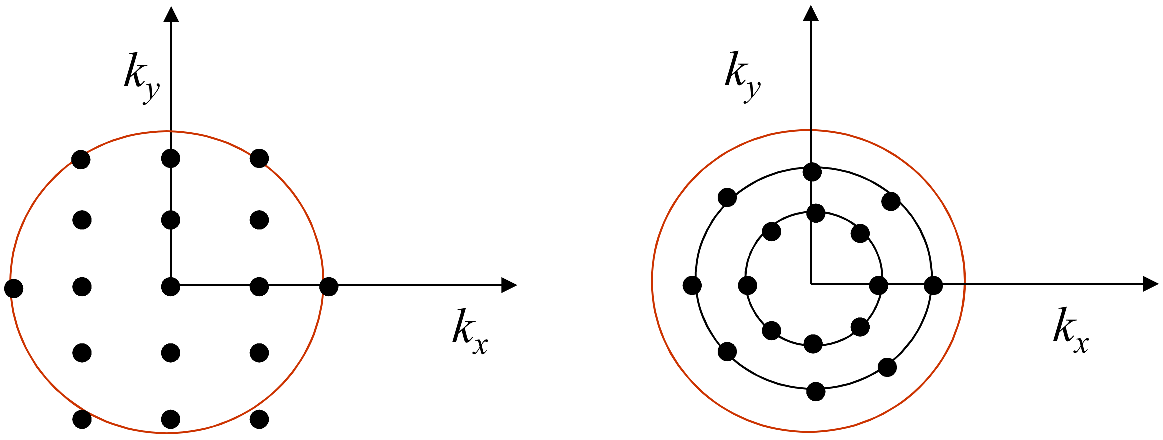


Circular motion is harmonic motion. Harmonic motion is quantized.

$$E_v = \hbar\omega_c\left(v+\tfrac{1}{2}\right) = \frac{\hbar^2}{2m}\left(k_x^2+k_y^2\right)$$

v labels the Landau level

Landau levels



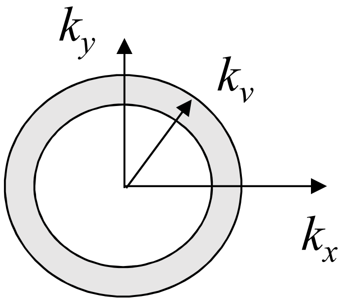
$B = 0$ $B \neq 0$

The global density of states in *k*-space does not change

Density of states 2D

$$E_v = \hbar\omega_c\left(v+\tfrac{1}{2}\right)$$

The number of states between ring *v*-1 and ring *v* is



$$\frac{\pi\left(k_v^2-k_{v-1}^2\right)}{\left(\frac{2\pi}{L}\right)^2}$$

$$\frac{\hbar^2k_v^2}{2m}=\hbar\omega_c\left(v+\tfrac{1}{2}\right)$$

$$k_v^2-k_{v-1}^2=\frac{2m\omega_c}{\hbar}\left[\left(v+\tfrac{1}{2}\right)-\left(v-1+\tfrac{1}{2}\right)\right]=\frac{2m\omega_c}{\hbar}$$

The number of states between ring *v*-1 and ring *v* is $\frac{m\omega_c}{2\pi\hbar}L^2$

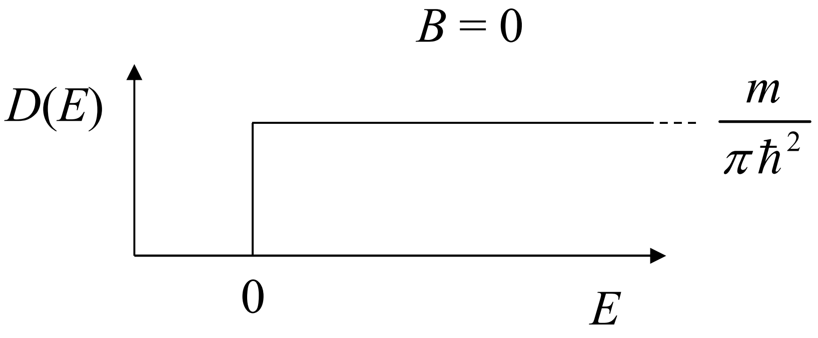
The density of states per spin is $\frac{m\omega_c}{2\pi\hbar}$

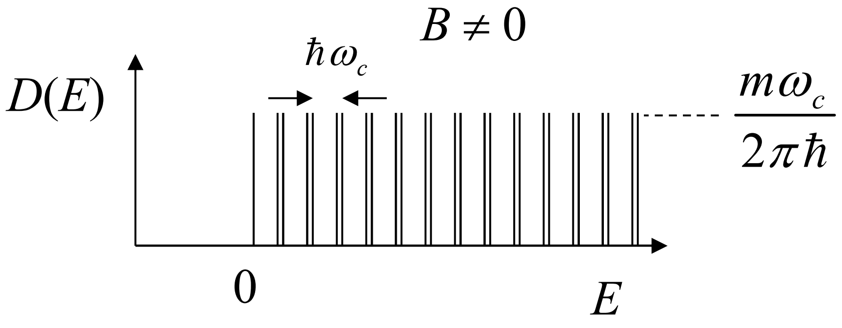
Spin

In a magnetic field, there is a shift of the energy of the electrons because of their spin.

$$E = -\vec{\mu} \cdot \vec{B} = \pm \frac{g}{2} \mu_B B$$

Bohr magneton $\mu_B = \frac{e\hbar}{2m_e}$ g-factor $g \approx 2$ $\hbar\omega_c = \frac{\hbar e B}{m} = 2\mu_B B$





$$D(E) = \frac{m\omega_c}{2\pi\hbar} \sum_{v=0}^{\infty} \delta\left(E - \hbar\omega_c\left(v + \tfrac{1}{2} - \tfrac{g}{4}\right)\right) + \delta\left(E - \hbar\omega_c\left(v + \tfrac{1}{2} + \tfrac{g}{4}\right)\right)$$