Harmonic oscillator

Newton's law: $ma = -Kx$

Euler - Lagrange equations: $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

Lagrangian (constructed by inspection) $$L(x, \dot{x}) = \frac{m\dot{x}^2}{2} - \frac{Kx^2}{2}$$

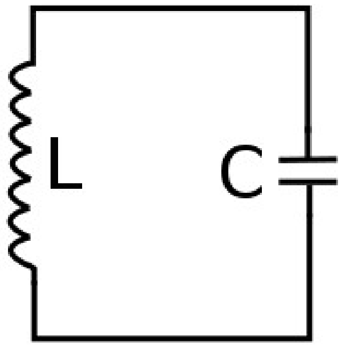
Conjugate variable: $$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

Legendre transformation: $$H = p\dot{x} - L = \frac{p^2}{2m} + \frac{Kx^2}{2}$$

Quantize: $p \rightarrow -i\hbar \frac{\partial}{\partial x}$ $$H\psi = \frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{Kx^2}{2}\psi$$

LC circuit

Classical equations $V = L\frac{dI}{dt}$ $I = -C\frac{dV}{dt}$ $Q = CV$

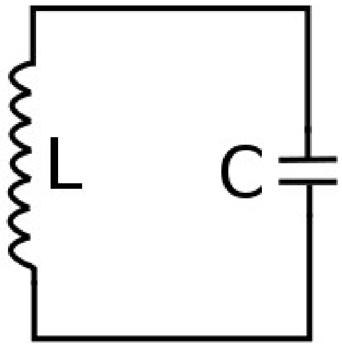


$$\frac{Q}{C} = -L\frac{d^2Q}{dt^2}$$

Euler - Lagrange equation: $$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Q}} - \frac{\partial \mathcal{L}}{\partial Q} = 0$$

Lagrangian (constructed by inspection) $$\mathcal{L}\left(Q,\dot{Q}\right) = \frac{L\dot{Q}^2}{2} - \frac{Q^2}{2C}$$

LC circuit



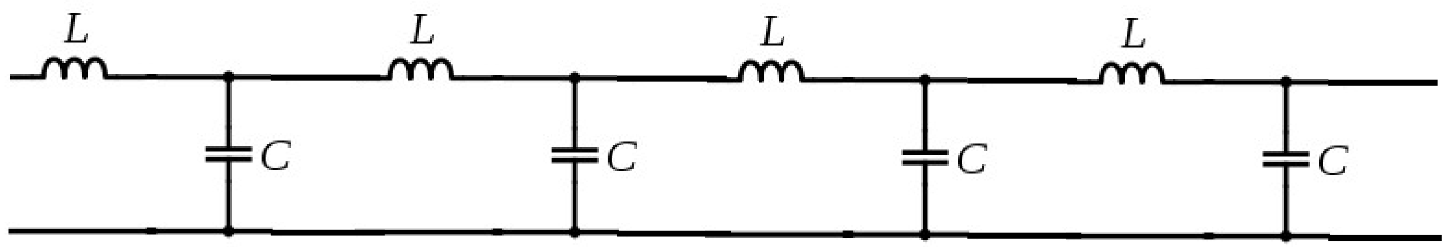
Conjugate variable: $$p = \frac{\partial \mathcal{L}}{\partial \dot{Q}} = L\dot{Q}$$

Legendre transformation: $$H = L\dot{Q}^2 - \mathcal{L} = \frac{L\dot{Q}^2}{2} + \frac{Q^2}{2C}$$

Quantize: $$p \to -i\hbar \frac{\partial}{\partial Q}$$

$$H\psi = \frac{-\hbar^2}{2L}\frac{d^2\psi}{dQ^2} + \frac{Q^2}{2C}\psi = E\psi$$

Transmission line



L inductance/m
C capacitance/m

$$-\frac{dV}{dx} = L\frac{dI}{dt} \qquad -\frac{dI}{dx} = C\frac{dV}{dt}$$

$$\frac{d^2V}{dx^2} = LC\frac{d^2V}{dt^2}$$

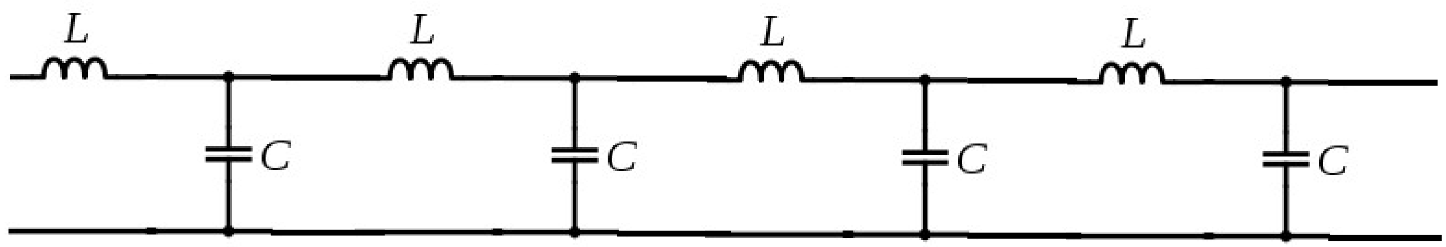
normal mode solution:

$$V_k = V_0 \exp\left(i\left(kx - \omega t\right)\right)$$

$$I_k = I_0 \exp\left(i\left(kx - \omega t\right)\right)$$

Each normal mode moves independently from the other normal modes

Transmission line



Substituting the normal mode solution $V = V_0 \exp\left(i\left(kx - \omega t\right)\right)$

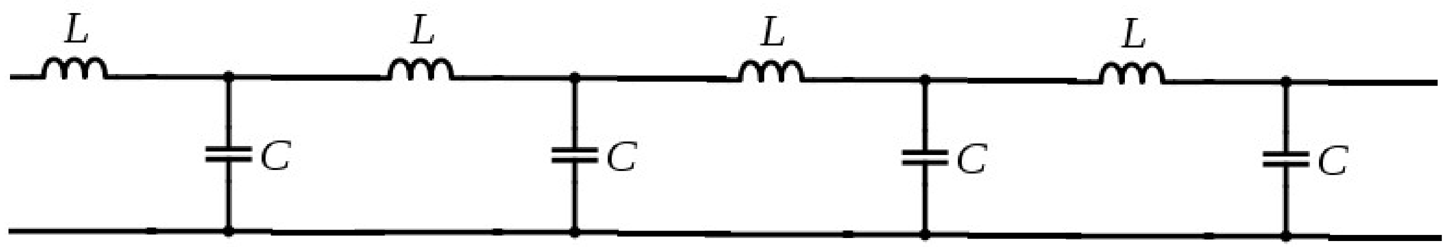
into the wave equation $\dfrac{d^2V}{dx^2} = LC\dfrac{d^2V}{dt^2} \longrightarrow -k^2 = -LC\omega^2$

yields the dispersion relation $\omega = \dfrac{k}{\sqrt{LC}} = ck$

$$I = \sqrt{\frac{C}{L}}V \qquad Z = \frac{V}{I} = \sqrt{\frac{L}{C}}$$

An infinite transmission line is resistive, typically ~ 50 Ω.

Transmission line



Wave equation

$$c^2 \frac{d^2V}{dx^2} = \frac{d^2V}{dt^2} \qquad c = \frac{1}{\sqrt{LC}}$$

c is the speed of waves

Not clear what mass we should use in the Schrödinger equation

The Schrödinger equation is for amateurs

Lagrangian (constructed by inspection) $\mathcal{L}(x,\dot{x})$ → Euler - Lagrange equations:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = 0$$

↓

Classical equations of motion (Newton's law)

↓

Conjugate variable: $p = \frac{\partial \mathcal{L}}{\partial \dot{x}}$

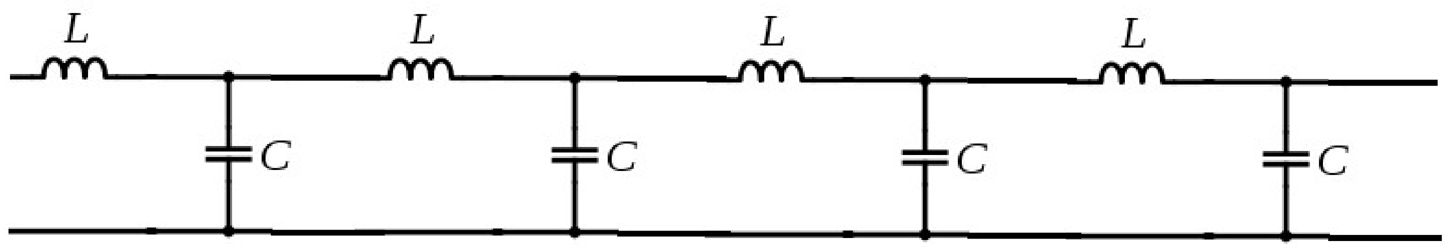
↓

Legendre transformation: $H = p\dot{x} - \mathcal{L}$

↓

Quantize: $p \rightarrow -i\hbar \frac{\partial}{\partial x}$

Transmission line



$$c^2 \frac{d^2V}{dx^2} = \frac{d^2V}{dt^2}$$

normal mode solution: $V_k = V_0 \exp\left(i\left(kx - \omega t\right)\right)$

$$-c^2k^2V_k = \frac{d^2V_k}{dt^2}$$

Each normal mode moves independently from the other normal modes

Lagrangian

Construct the Lagrangian 'by inspection'. The Euler-Lagrange equation and the classical equation of motion for a normal mode are,

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{V}_k}\right) - \frac{\partial \mathcal{L}}{\partial V_k} = 0. \qquad -c^2 k^2 V_k = \frac{\partial^2 V_k}{\partial t^2}.$$

↑ classical equation for the mode *k*

The Lagrangian is, $$\mathcal{L} = \frac{\dot{V}_k^{\,2}}{2} - \frac{c^2 k^2}{2} V_k^{\,2}$$

Hamiltonian

$$\mathcal{L} = \frac{\dot{V}_k^{\,2}}{2} - \frac{c^2k^2}{2}V_k^{\,2}$$

The conjugate variable to V_k is,

$$\frac{\partial \mathcal{L}}{\partial \dot{V}_k} = \dot{V}_k$$

The Hamiltonian is constructed by performing a Legendre transformation,

$$H = \dot{V}_k\dot{V}_k - \mathcal{L} = \frac{\dot{V}_k^{\,2}}{2} + \frac{c^2k^2}{2}V_k^{\,2}$$

To quantize we replace the conjugate variable by $-i\hbar\frac{\partial}{\partial V_k}$

$$\frac{-\hbar^2}{2}\frac{d^2\psi}{dV_k^{\,2}} + \frac{c^2k^2}{2}V_k^{\,2}\psi = E\psi$$

Quantum solutions

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{K}{2}x^2\psi=E\psi \qquad \frac{-\hbar^2}{2}\frac{d^2\psi}{d{V_k}^2}+\frac{c^2k^2}{2}{V_k}^2\psi=E\psi$$

This equation is mathematically equivalent to the harmonic oscillator.

$$E=\hbar\omega\left(j+\tfrac{1}{2}\right) \qquad j=0,1,2,\ldots$$

spring constant →

$$\omega=\sqrt{\frac{K}{m}}$$

mass - spring

$$\omega=\sqrt{c^2k^2}$$

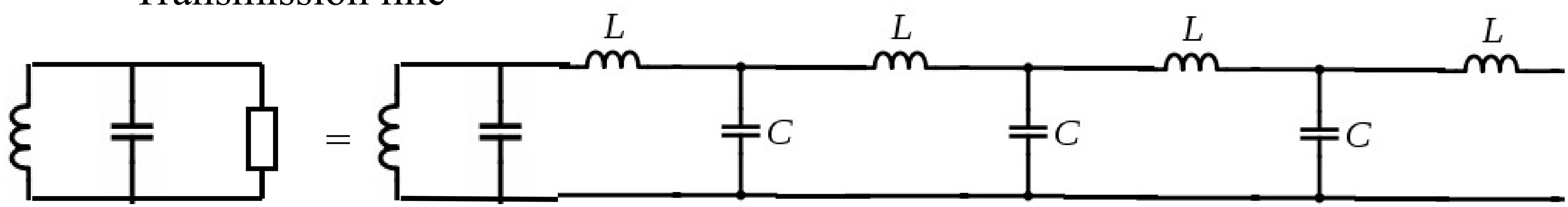
wave mode

$$\omega=c\left|\vec{k}\right|$$

j is the number of photons.

Dissipation in Quantum mechanics

Transmission line

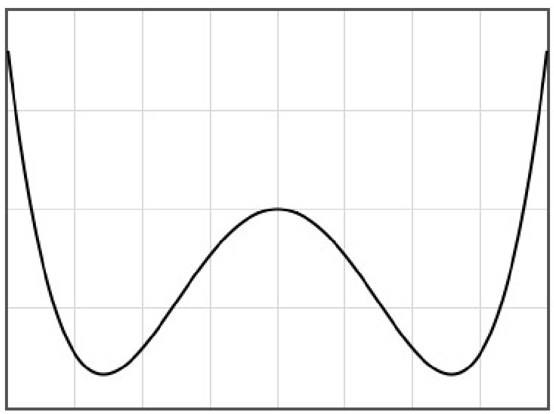


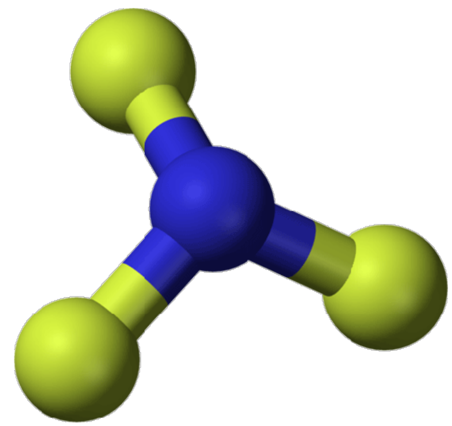
$$I = \sqrt{\frac{C}{L}} V$$

$$Z = \frac{V}{I} = \sqrt{\frac{L}{C}}$$

An infinite transmission line is resistive

Nitrogen



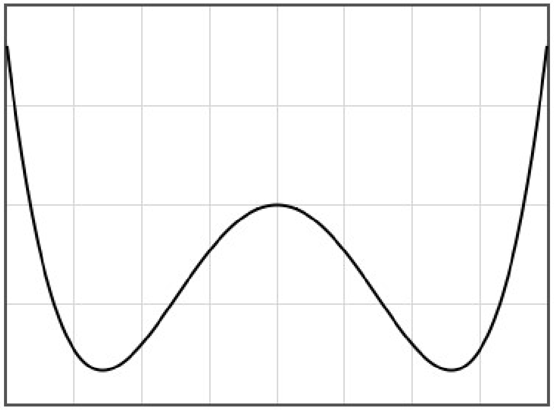


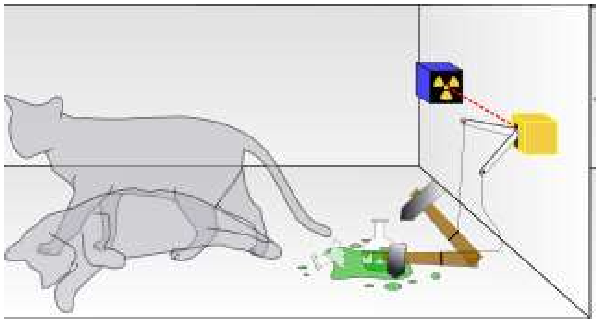
$$\psi_0 \propto \exp\left(-\frac{iE_0 t}{\hbar}\right)$$

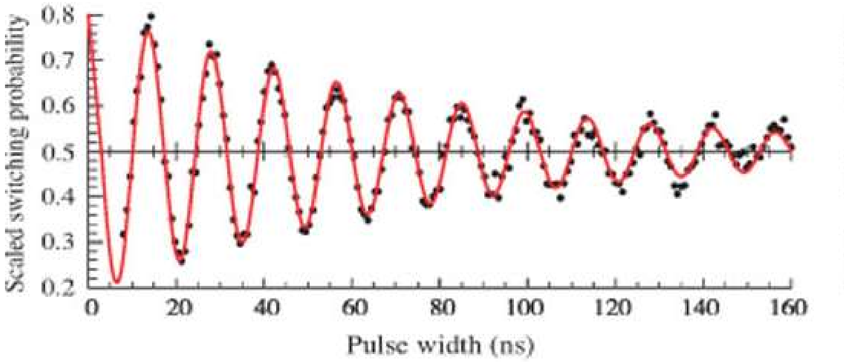
$$\psi_1 \propto \exp\left(-\frac{iE_1 t}{\hbar}\right)$$

Dissipation in quantum mechanics

Quantum coherence is maintained until the decoherence time. This depends on the strength of the coupling of the quantum system to other







Decay is the decoherence time.

Dissipation in solids

At zero electric field, the electron eigen states are Bloch states. Each Bloch state has a k vector. The average value of $k = 0$ (no current).

At finite electric field, the Bloch states are no longer eigen states but we can calculate the transitions between Bloch states using Fermi's golden rule. The final state may include an electron state plus a phonon. The average value of k is not zero (finite current).

The phonons carry the energy away like a transmission line.

The quantization of the electromagnetic field

Wave nature and the particle nature of light

Unification of the laws for electricity and magnetism (described by Maxwell's equations) and light

Quantization of fields

Derive the Bose-Einstein function

Planck's radiation law

Serves as a template for the quantization of noninteracting bosons: phonons, magnons, plasmons, and other quantum particles that inhabit solids.

http://lamp.tu-graz.ac.at/~hadley/ss2/emfield/quantization_em.php

Maxwell's equations

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

The vector potential

$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

Maxwell's equations in terms of A

Coulomb gauge $\nabla \cdot \vec{A} = 0$

$$\nabla \cdot \frac{\partial \vec{A}}{\partial t} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t} \nabla \cdot \vec{A} = 0$$

$$\nabla \cdot \nabla \times \vec{A} = 0 \quad \Rightarrow \quad \text{Vector identity}$$

$$\nabla \times \frac{\partial \vec{A}}{\partial t} = \frac{\partial}{\partial t} \nabla \times \vec{A} \quad \Rightarrow \quad \frac{\partial}{\partial t} \nabla \times \vec{A} = \frac{\partial}{\partial t} \nabla \times \vec{A}$$

$$\nabla \times \nabla \times \vec{A} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$$

The wave equation

$$\nabla \times \nabla \times \vec{A} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$$

Using the identity $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A},$

wave equation

$$\boxed{c^2 \nabla^2 \vec{A} = \frac{\partial^2 \vec{A}}{\partial t^2}.}$$

normal mode solutions have the form: $\vec{A}(\vec{r}, t) = \vec{A} \exp(i(\vec{k} \cdot \vec{r} - \omega t))$

Substituting the normal mode solution in the wave equation results in the dispersion relation

$$\omega = c\left|\vec{k}\right|$$

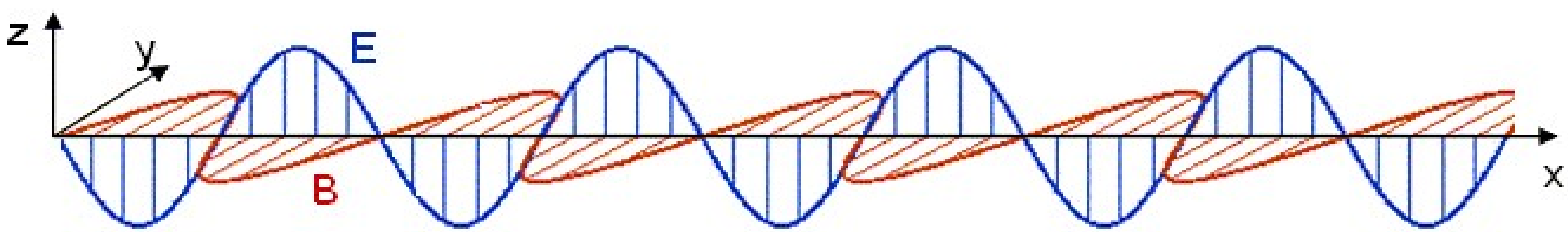
EM waves propagating in the x direction

$$\vec{A} = A_0 \cos(k_x x - \omega t)\hat{z}$$

The electric and magnetic fields are

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\omega A_0 \sin(k_x x - \omega t)\hat{z},$$

$$\vec{B} = \nabla \times \vec{A} = k_x A_0 \sin(k_x x - \omega t)\hat{y}.$$



Lagrangian

To quantize the wave equation we first construct the Lagrangian 'by inspection'. The Euler-Lagrange equation and the classical equation of motion are,

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{A}_s}\right) - \frac{\partial L}{\partial A_s} = 0. \qquad -c^2 k^2 A_s = \frac{\partial^2 A_s}{\partial t^2}.$$

classical equation for the normal mode k

The Lagrangian is, $$L = \frac{\dot{A}_s^2}{2} - \frac{c^2 k^2}{2} A_s^2$$

Hamiltonian

$$L = \frac{\dot{A}_s^2}{2} - \frac{c^2 k^2}{2} A_s^2$$

The conjugate variable to A_s is,

$$\frac{\partial L}{\partial \dot{A}_s} = \dot{A}_s$$

The Hamiltonian is constructed by performing a Legendre transformation,

$$H = \dot{A}_s \dot{A}_s - L = \frac{\dot{A}_s^2}{2} + \frac{c^2 k^2}{2} A_s^2$$

To quantize we replace the conjugate variable by $-i\hbar \frac{\partial}{\partial A_s}$

$$\frac{-\hbar^2}{2} \frac{d^2 \psi}{d A_s^2} + \frac{c^2 k^2}{2} A_s^2 \psi = E\psi$$

Quantum solutions

$$\frac{-\hbar^2}{2}\frac{d^2\psi}{dA_s^2}+\frac{c^2k^2}{2}A_s^2\psi = E\psi$$

This equation is mathematically equivalent to the harmonic oscillator.

$$E_s = \hbar\omega_s\left(j_s + \tfrac{1}{2}\right) \qquad j_s = 0,1,2,\ldots$$

$$\omega_s = c\left|\vec{k}_s\right|$$

j_s is the number of photons in mode s.

Non-interacting boson systems

Photons, phonons, magnons, plasmons can be approximated as non-interacting bosons.

To calculate their thermodynamic properties:

Construct the partition function

$$Z_{gr}(T,\mu)=\sum_q \exp\left(\frac{\mu}{k_BT}\right)^{N_q} \exp\left(-\frac{E_q}{k_BT}\right)=\sum_q \exp\left(-\frac{E_q-N_q\mu}{k_BT}\right)^{N_q}$$

$$\phi=-\frac{k_BT}{V}\ln(Z_{gr})$$

Deduce the thermodynamic properties:

$$n=-\frac{\partial\phi}{\partial\mu} \qquad f=\phi+n\mu \qquad s=-\frac{\partial\phi}{\partial T}$$