The quantization of the electromagnetic field

Wave nature and the particle nature of light

Unification of the laws for electricity and magnetism (described by Maxwell's equations) and light

Quantization of fields

Derive the Bose-Einstein function

Planck's radiation law

Serves as a template for the quantization of noninteracting bosons: phonons, magnons, plasmons, and other quantum particles that inhabit solids.

http://lamp.tu-graz.ac.at/~hadley/ss2/emfield/quantization_em.php

Maxwell's equations

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

The vector potential

$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

Maxwell's equations in terms of A

Coulomb gauge $\nabla \cdot \vec{A} = 0$

$$\nabla \cdot \frac{\partial \vec{A}}{\partial t} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t} \nabla \cdot \vec{A} = 0$$

$$\nabla \cdot \nabla \times \vec{A} = 0 \quad \Rightarrow \quad \text{Vector identity}$$

$$\nabla \times \frac{\partial \vec{A}}{\partial t} = \frac{\partial}{\partial t} \nabla \times \vec{A} \quad \Rightarrow \quad \frac{\partial}{\partial t} \nabla \times \vec{A} = \frac{\partial}{\partial t} \nabla \times \vec{A}$$

$$\nabla \times \nabla \times \vec{A} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$$

The wave equation

$$\nabla \times \nabla \times \vec{A} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$$

Using the identity $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A},$

wave equation

$$c^2 \nabla^2 \vec{A} = \frac{\partial^2 \vec{A}}{\partial t^2}.$$

normal mode solutions have the form: $\vec{A}(\vec{r}, t) = \vec{A} \exp(i(\vec{k} \cdot \vec{r} - \omega t))$

Substituting the normal mode solution in the wave equation results in the dispersion relation

$$\omega = c\left|\vec{k}\right|$$

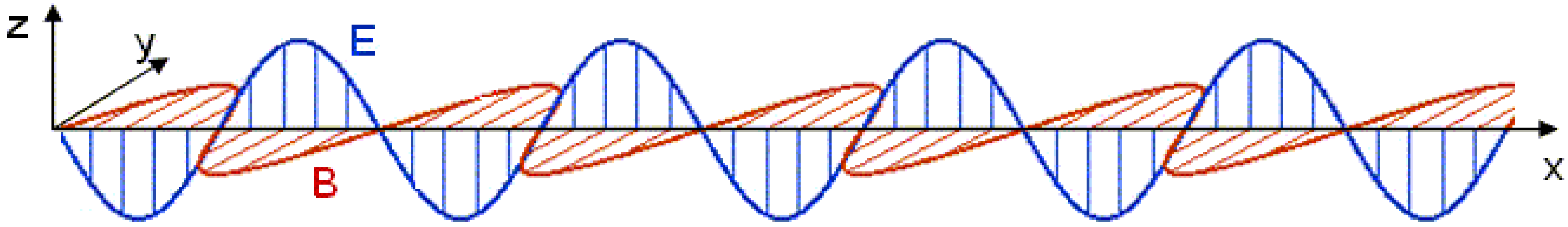
EM waves propagating in the x direction

$$\vec{A} = A_0 \cos(k_x x - \omega t)\hat{z}$$

The electric and magnetic fields are

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\omega A_0 \sin(k_x x - \omega t)\hat{z},$$

$$\vec{B} = \nabla \times \vec{A} = k_x A_0 \sin(k_x x - \omega t)\hat{y}.$$



Lagrangian

To quantize the wave equation we first construct the Lagrangian 'by inspection'. The Euler-Lagrange equation and the classical equation of motion are,

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{A}_s}\right) - \frac{\partial L}{\partial A_s} = 0. \qquad -c^2 k^2 A_s = \frac{\partial^2 A_s}{\partial t^2}.$$

classical equation for the normal mode *k*

The Lagrangian is, $$L = \frac{\dot{A}_s^2}{2} - \frac{c^2 k^2}{2} A_s^2$$

Hamiltonian

$$L = \frac{\dot{A}_s^2}{2} - \frac{c^2 k^2}{2} A_s^2$$

The conjugate variable to A_s is,

$$\frac{\partial L}{\partial \dot{A}_s} = \dot{A}_s$$

The Hamiltonian is constructed by performing a Legendre transformation,

$$H = \dot{A}_s \dot{A}_s - L = \frac{\dot{A}_s^2}{2} + \frac{c^2 k^2}{2} A_s^2$$

To quantize we replace the conjugate variable by $-i\hbar \frac{\partial}{\partial A_s}$

$$\frac{-\hbar^2}{2} \frac{d^2 \psi}{dA_s^2} + \frac{c^2 k^2}{2} A_s^2 \psi = E\psi$$

Quantum solutions

$$\frac{-\hbar^2}{2}\frac{d^2\psi}{dA_s^2}+\frac{c^2k^2}{2}A_s^2\psi=E\psi$$

This equation is mathematically equivalent to the harmonic oscillator.

$$E_s=\hbar\omega_s\left(j_s+\tfrac{1}{2}\right) \qquad j_s=0,1,2,\ldots$$

$$\omega_s=c\left|\vec{k}_s\right|$$

j_s is the number of photons in mode s.

Thermodynamic properties of non-interacting bosons

The grand canonical partition function is

$$Z_{gr} = \sum_q \exp\left(\frac{\mu}{k_B T}\right)^{N_q} \exp\left(-\frac{E_q}{k_B T}\right) = \sum_q \exp\left(-\frac{E_q - \mu N_q}{k_B T}\right)$$

Here q sums over the macro states. Any number of bosons can occupy a microscopic quantum state.

$$N_q = \sum_i n_{qi} \qquad E_q = \sum_i n_{qi}\varepsilon_i \qquad n_{qi} \in 0,1,2,\cdots,\infty$$

n_{qi} are occupation numbers that specify if microstate i is occupied in macrostate q

http://lampx.tugraz.at/~hadley/ss2/emfield/thermo/thermo.php

Thermodynamic properties of non-interacting bosons

$$Z_{gr} = \sum_q \exp\left(-\frac{E_q - \mu N_q}{k_B T}\right) = \sum_q \exp\left(-\frac{\sum_i n_{qi}(\varepsilon_i - \mu)}{k_B T}\right) = \sum_q \prod_i \exp\left(-\frac{n_{qi}(\varepsilon_i - \mu)}{k_B T}\right)$$

The sum over all possible macrostates can also be written as the sum over all possible microstates.

$$Z_{gr} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_{\max}=0}^{\infty} \prod_i \exp\left(-\frac{n_i(\varepsilon_i - \mu)}{k_B T}\right)$$

Pull the n_i factors through the other sums..

$$Z_{gr} = \left[\sum_{n_1=0}^{\infty} \exp\left(-\frac{n_1(\varepsilon_1 - \mu)}{k_B T}\right)\right]\left[\sum_{n_2=0}^{\infty} \exp\left(-\frac{n_2(\varepsilon_2 - \mu)}{k_B T}\right)\right] \cdots \left[\sum_{n_j}^{\infty} \exp\left(-\frac{n_j(\varepsilon_j - \mu)}{k_B T}\right)\right] \cdots$$

Thermodynamic properties of non-interacting bosons

$$Z_{gr} = \left[\sum_{n_1=0}^{\infty} \exp\left(-\frac{n_1(\varepsilon_1-\mu)}{k_B T}\right)\right]\left[\sum_{n_2=0}^{\infty} \exp\left(-\frac{n_2(\varepsilon_2-\mu)}{k_B T}\right)\right]\cdots\left[\sum_{n_j}^{\infty} \exp\left(-\frac{n_j(\varepsilon_j-\mu)}{k_B T}\right)\right]\cdots$$

All the products sum over n_i so we can relabel to sum over n.

$$Z_{gr} = \prod_i \left[\sum_{n=0}^{\infty} \exp\left(-\frac{n(\varepsilon_i-\mu)}{k_B T}\right)\right]$$

The sum is

$$\sum_{n=0}^{\infty} \exp\left(-\frac{n(\varepsilon_i-\mu)}{k_B T}\right) = 1 + x + x^2 + \cdots \qquad x = \exp\left(-\frac{(\varepsilon_i-\mu)}{k_B T}\right)$$

If $x < 1$, this is a geometric series $\quad 1 + x + x^2 + \cdots = \dfrac{1}{1-x}$

Thermodynamic properties of non-interacting bosons

$$Z_{gr} = \prod_i \left[1 - \exp\left(-\frac{\varepsilon_i - \mu}{k_B T}\right)\right]^{-1}$$

Grand potential: $\Phi = U - TS - \mu N = -k_B T \ln(Z_{gr})$

$$\Phi = -k_B T \ln(Z_{gr}) = k_B T \sum_i \left[1 - \exp\left(\frac{\mu - \varepsilon_i}{k_B T}\right)\right]$$

$$n = -\frac{\partial \phi}{\partial \mu} = k_B T \int_{-\infty}^{\infty} D(E) \frac{1}{1 - \exp\left(\frac{\mu - E}{k_B T}\right)} \exp\left(\frac{\mu - E}{k_B T}\right) \cdot \frac{1}{k_B T} \, dE = \int_{-\infty}^{\infty} D(E) \underbrace{\frac{1}{\exp\left(\frac{E-\mu}{k_B T}\right) - 1}}_{F_{BE}(E)} dE$$

Thermodynamic properties of non-interacting bosons

$$f = \phi + \mu n = \int_{-\infty}^{\infty} D(E) \left\{ k_B T \ln\left[1 - \exp\left(\frac{\mu - E}{k_B T}\right)\right] + \frac{\mu}{\exp\left(\frac{E-\mu}{k_B T}\right) - 1} \right\} dE$$

$$s = -\frac{\partial \phi}{\partial T} = \frac{1}{T} \int_{-\infty}^{\infty} D(E) \left\{ -k_B T \ln\left[1 - \exp\left(\frac{\mu - E}{k_B T}\right)\right] + \frac{E - \mu}{\exp\left(\frac{E-\mu}{k_B T}\right) - 1} \right\} dE$$

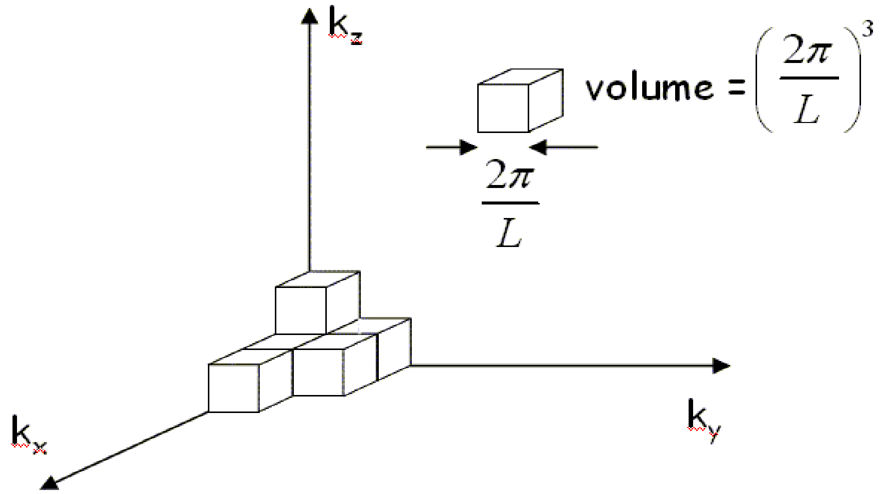
$$u = f + Ts = \int_{-\infty}^{\infty} \frac{E D(E)}{\exp\left(\frac{E-\mu}{k_B T}\right) - 1} dE$$

$$c_v = \frac{\partial u}{\partial T} = \int_{-\infty}^{\infty} \frac{E D(E)(E - \mu) \exp\left(\frac{E-\mu}{k_B T}\right)}{k_B T^2 \left[\exp\left(\frac{E-\mu}{k_B T}\right) - 1\right]^2} dE$$

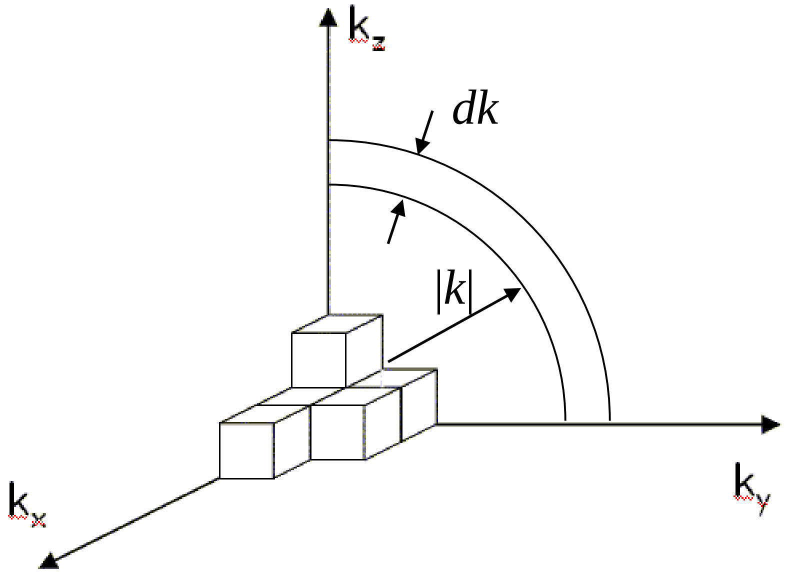
Normal modes

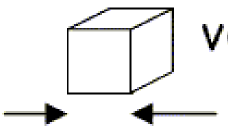
For an electromagnetic field in a cubic region of length L with periodic boundary conditions, the vector potential has the form:

$$\vec{A}_{\perp} \sin\left(k_x x + k_y y + k_z z\right) \qquad k_x, k_y, k_z = \ldots -\frac{4\pi}{L}, -\frac{2\pi}{L}, 0, \frac{2\pi}{L}, \frac{4\pi}{L} \ldots$$



Density of states





volume = $\left(\frac{2\pi}{L}\right)^3$

$\frac{2\pi}{L}$

$$k_x, k_y, k_z = \ldots, \frac{-4\pi}{L}, \frac{-2\pi}{L}, 0, \frac{2\pi}{L}, \frac{4\pi}{L}, \ldots$$

$$L^3 D(k)dk = 2\frac{4\pi k^2 dk}{\left(\frac{2\pi}{L}\right)^3} = \frac{k^2 L^3}{\pi^2}dk$$

polarizations

$$D(k) = \frac{k^2}{\pi^2}$$

Density of states

$$D(k)dk = \frac{k^2}{\pi^2}dk = D(\omega)d\omega$$

use the dispersion relation to convert $D(k)$ to $D(\omega)$

$$\omega = ck$$

$$d\omega = cdk$$

$$D(\omega) = \frac{\omega^2}{c^3\pi^2}$$

$$D(E) = \frac{E^2}{\pi^2 c^3 \hbar^3}$$

Thermodynamic properties of non-interacting bosons

The number of photons is not conserved.

$$\left.\frac{\partial F}{\partial N}\right|_{T,V} = 0 = \mu.$$

- **Photons**
  - Thermodynamic properties of non-interacting bosons
    - Particle density n
    - Grand potential φ
    - Helmholtz free energy $f(T)$
    - Entropy $s(T)$
    - Energy spectral $u(\omega,T)$
    - Internal energy $u(T)$
    - Specific heat $c_v(T)$

Summary of the results for the quantization of the wave equation in 1,2, and 3 dimensions

	1-D	2-D
Wave Equation c = speed of light A_j = j^{th} component of the vector potential	$c^2\frac{d^2A_j}{dx^2}=\frac{d^2A_j}{dt^2}$	$c^2\left(\frac{d^2A_j}{dx^2}+\frac{d^2A_j}{dy^2}\right)=\frac{d^2A_j}{dt^2}$
Eigenfunction solutions k = wavenumber ω = angular frequency	$A_j=\exp\left(i\left(kx-\omega t\right)\right)$	$A_j=\exp\left(i\left(\vec{k}\cdot\vec{r}-\omega t\right)\right)$
Dispersion relation	$\omega=ck$	$\omega=c\left\|\vec{k}\right\|$
Density of states	$D(k)=\frac{2}{\pi}$	$D(k)=\frac{k}{\pi}$ [m^{-1}]
Density of states $D(\omega)=D(k)\frac{dk}{d\omega}$	$D(\omega)=\frac{2}{\pi c}$ [s/m]	$D(\omega)=\frac{\omega}{\pi c^2}$ [s/m^2]
Density of states $D(\lambda)=D(k)\frac{dk}{d\lambda}$ λ = wavelength	$D(\lambda)=\frac{4}{\lambda^2}$ [m^{-2}]	$D(\lambda)=\frac{4\pi}{\lambda^3}$ [m^{-3}]
Density of states $D(E)=D(\omega)\frac{d\omega}{dE}$	$D(E)=\frac{2}{\pi\hbar c}$ [J^{-1}m^{-1}]	$D(E)=\frac{E}{\pi\hbar^2c^2}$ [J^{-1}m^{-2}]
Chemical potential	$\mu = 0$	$\mu = 0$
Intensity spectral density k_B = 1.3806504 × 10^{-23} [J/K] Boltzmann's constant h = 6.62606896 × 10^{-34} [J s] Planck's constant	$I(\lambda)=\frac{2hc^2}{\lambda^3\left(\exp\left(\frac{hc}{\lambda k_BT}\right)-1\right)}$ [J m^{-1}s^{-1}]	$I(\lambda)=\frac{4hc^2}{\lambda^4\left(\exp\left(\frac{hc}{\lambda k_BT}\right)-1\right)}$ [J m^{-2}s^{-1}]
Wien's law $\left.\frac{dI(\lambda)}{d\lambda}\right\|_{\lambda=\lambda_{max}}=0$	$\lambda_{max}=\frac{0.0050994367}{T}$ [m]	$\lambda_{max}=\frac{0.0036696984}{T}$ [m]
Stefan - Boltzmann law $I=\int_0^\infty I(\lambda)d\lambda$ $\zeta(3)\approx 1.202$ Riemann ζ function σ = 5.67 × 10^{-8} Stefan-Boltzmann constant	$I=\frac{\pi^2k_B{}^2T^2}{3h}$ [J s^{-1}]	$I=\frac{8\varsigma(3)k_B{}^3T^3}{h^2c}$ [J m^{-1} s^{-1}]