Electrons in a magnetic field

Technische Universität Graz

Charged particle in a magnetic field

$$\vec{F} = -e\vec{v} \times \vec{B} = ma$$

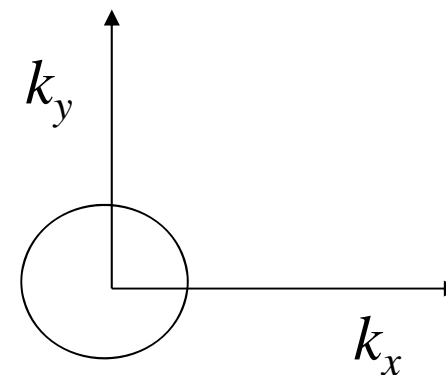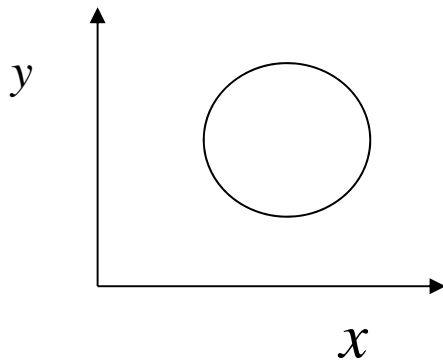$$evB_z = \frac{mv^2}{R}$$

Magnetron

solve for velocity

$$v = \frac{eB_z R}{m}$$

$$v = \omega_c R$$
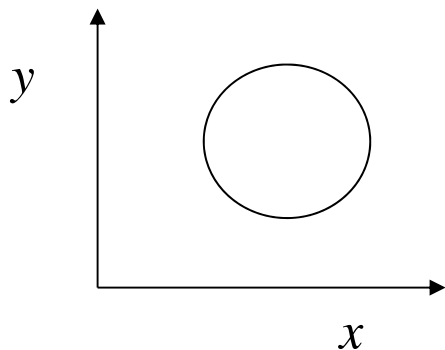
$$\omega_c = \frac{eB_z}{m}$$

y

x

k_y

k_x

Why don't metals in a B field radiate?
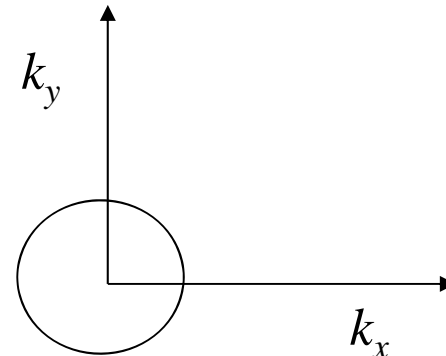
Magnetron

$$\omega_c = \frac{eB_z}{m}$$

There are no lower lying states to fall into. We need a quantum description.

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

Magnetic force depends on the velocity, not on the position.

Electrons in a magnetic field

Lorentz force law

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

Euler Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

Lagrangian:

$$L = \tfrac{1}{2}mv^2 - qV\left(\vec{r},t\right) + q\vec{v}\cdot\vec{A}\left(\vec{r},t\right)$$

Kittel: Appendix G
http://lamp.tu-graz.ac.at/~hadley/ss2/IQHE/cpimf.php

Lagrangian

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 \qquad\qquad L = \tfrac{1}{2}mv^2 - qV\left(\vec{r},t\right) + q\vec{v}\cdot\vec{A}\left(\vec{r},t\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_x}\right) = \frac{d}{dt}\left(mv_x + qA_x\right) = m\frac{dv_x}{dt} + q\frac{dA_x}{dt}$$

$$= m\frac{dv_x}{dt} + q\left(\frac{dx}{dt}\frac{\partial A_x}{\partial x} + \frac{dy}{dt}\frac{\partial A_x}{\partial y} + \frac{dz}{dt}\frac{\partial A_x}{\partial z} + \frac{\partial A_x}{\partial t}\right)$$

$$= m\frac{dv_x}{dt} + q\left(v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z} + \frac{\partial A_x}{\partial t}\right),$$

$$\frac{\partial L}{\partial x} = -q\frac{\partial V}{\partial x} + q\left(v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x}\right)$$

$$m\frac{dv_x}{dt} = -q\left(\frac{\partial V}{\partial x} + \frac{\partial A_x}{\partial t}\right) + q\left(v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + v_z\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\right)$$

Lagrangian

$$m\frac{dv_x}{dt} = -q\left(\frac{\partial V}{\partial x} + \frac{\partial A_x}{\partial t}\right) + q\left(v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + v_z\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\right)$$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \qquad \text{and} \qquad \vec{B} = \nabla \times \vec{A}$$

$$m\frac{dv_x}{dt} = q\left(E_x + \left(\vec{v} \times \vec{B}\right)_x\right)$$

$$L = \tfrac{1}{2}mv^2 - qV(\vec{r},t) + q\vec{v} \cdot \vec{A}(\vec{r},t)$$

conjugate variable: $\qquad p_x = \dfrac{\partial L}{\partial v_x} = mv_x + qA_x$

kinetic momentum

field momentum
(inductance)

$$v_x = \frac{1}{m}\left(p_x - qA_x\right)$$

Hamiltonian

$$v_x = \frac{1}{m}\left(p_x - qA_x\right)$$

$$L = \tfrac{1}{2}mv^2 - qV\left(\vec{r},t\right) + q\vec{v}\cdot\vec{A}\left(\vec{r},t\right)$$

Legendre transformation
$$H = \vec{v}\cdot\vec{p} - L$$

$$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + qV$$

Classical result

$$\vec{p} \rightarrow -i\hbar\nabla$$

Schrödinger equation
$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(-i\hbar\nabla - q\vec{A}\right)^2\psi + qV\psi$$

Landau Levels

free particles in a magnetic field

$$\frac{1}{2m}\left(-i\hbar\nabla - q\vec{A}\right)^2 \psi(\vec{r}) = E\psi(\vec{r}). \qquad V = 0$$

Landau gauge $\qquad \vec{A} = B_z x\hat{y}.$

$$\vec{B} = \nabla \times \vec{A} = B_z x\hat{y} = \left(\frac{dA_z}{dy} - \frac{dA_y}{dz}\right)\hat{x} + \left(\frac{dA_x}{dz} - \frac{dA_z}{dx}\right)\hat{y} + \left(\frac{dA_y}{dx} - \frac{dA_x}{dy}\right)\hat{z}.$$

$$\vec{B} = B_z\hat{z}.$$

Landau Levels

free particles in a magnetic field

$$\frac{1}{2m}\left(-i\hbar\nabla - q\vec{A}\right)^2 \psi(\vec{r}) = E\psi(\vec{r}). \qquad V = 0$$

Landau gauge $\quad \vec{A} = B_z x\hat{y}.$

$$\left(-i\hbar\nabla - q\vec{A}\right)^2 = \left(-i\hbar\nabla - qB_z x\hat{y}\right)\cdot\left(-i\hbar\nabla - qB_z x\hat{y}\right)$$

$$-i\hbar\nabla\cdot\left(-qB_z x\hat{y}\right) = -i\hbar\left(\frac{d}{dx}\hat{x} + \frac{d}{dy}\hat{y} + \frac{d}{dz}\hat{z}\right)\cdot\left(-qB_z x\hat{y}\right) = i\hbar qB_z x\frac{d}{dy}$$

$$\frac{1}{2m}\left(-\hbar^2\nabla^2 + i2\hbar qB_z x\frac{d}{dy} + q^2 B_z^2 x^2\right)\psi = E\psi.$$

The solution has the form

$$\psi = e^{ik_y y}e^{ik_z z}\phi(x).$$

Landau Levels

$$\frac{1}{2m}\left(-\hbar^2\nabla^2 + i2\hbar qB_z x\frac{d}{dy} + q^2B_z^2 x^2\right)\psi = E\psi.$$

The solution has the form

$$\psi = e^{ik_y y}e^{ik_z z}\phi(x).$$

Substitute this into the equation

$$\frac{1}{2m}\left(-\hbar^2\frac{\partial^2}{\partial x^2} + \hbar^2 k_z^2 + \hbar^2 k_y^2 - 2\hbar qB_z k_y x + q^2 B_z^2 x^2\right)\phi(x) = E\phi(x).$$

Landau Levels

The equation for $\phi(x)$ is

$$\frac{1}{2m}\left(-\hbar^2\frac{\partial^2}{\partial x^2} + \hbar^2 k_z^2 + \underbrace{\hbar^2 k_y^2 - 2\hbar q B_z k_y x + q^2 B_z^2 x^2}_{\left(\hbar k_y - q B_z x\right)^2}\right)\phi(x) = E\phi(x).$$

$$\frac{1}{2m}\left(-\hbar^2\frac{\partial^2}{\partial x^2} + q^2 B_z^2\left(x - \frac{\hbar k_y}{q B_z}\right)^2\right)\phi(x) = \left(E - \frac{\hbar^2 k_z^2}{2m}\right)\phi(x).$$

This is the equation for a harmonic oscillator

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{K}{2}\left(x - x_0\right)^2\right)\phi(x) = E'\phi(x).$$

Landau Levels

$$\frac{1}{2m}\left(-\hbar^2\frac{\partial^2}{\partial x^2}+q^2B_z^2\left(x-x_0\right)^2\right)\phi(x)=\left(E-\frac{\hbar^2 k_z^2}{2m}\right)\phi(x). \quad x_0=\frac{\hbar k_y}{qB_z}$$

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{K}{2}\left(x-x_0\right)^2\right)\phi(x)=E'\phi(x).$$

This is the equation for a harmonic oscillator

$$\frac{K}{2}\Leftrightarrow\frac{q^2B_z^2}{2m}$$

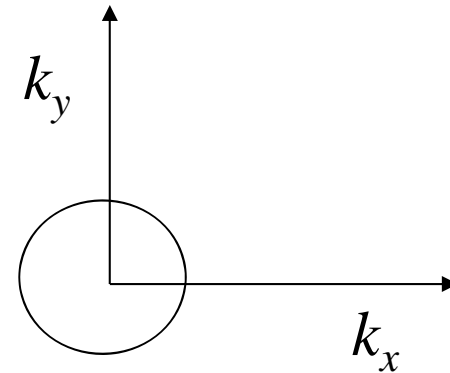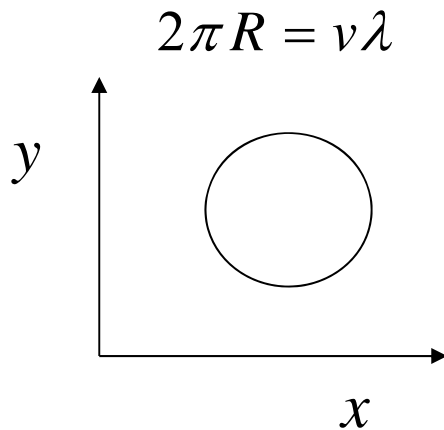$$\omega_c=\sqrt{\frac{K}{m}}\Leftrightarrow\frac{qB_z}{m}$$

$$E_{k_z,v}=\frac{\hbar^2 k_z^2}{2m}+\hbar\omega_c\left(v+\tfrac{1}{2}\right) \qquad \omega_c=\frac{qB_z}{m}$$

Charged particle in a magnetic field

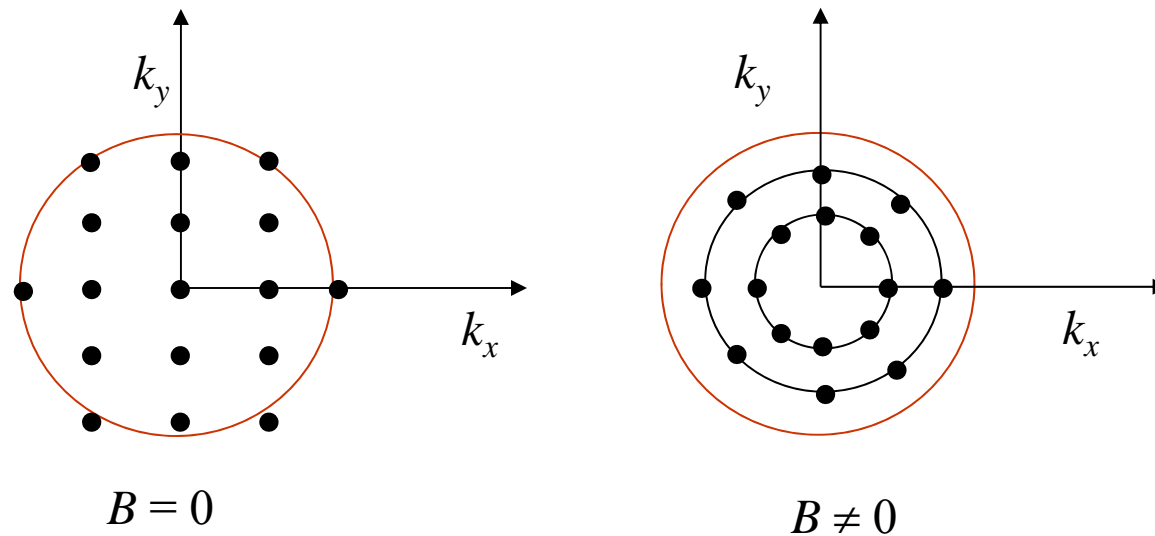Bohr - Sommerfeld quantization

$$2\pi R = v\lambda$$

Circular motion is harmonic motion. Harmonic motion is quantized.

$$E_v = \hbar\omega_c\left(v + \tfrac{1}{2}\right) = \frac{\hbar^2}{2m}\left(k_x^2 + k_y^2\right)$$

v labels the Landau level

Landau levels

$B = 0$

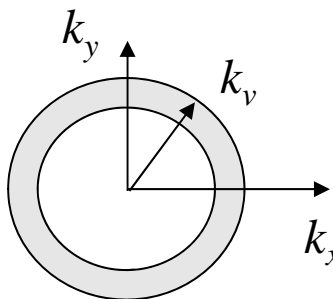$B \neq 0$

The global density of states in *k*-space does not change

In 2-D, the k-volume per k state is: $\left(\dfrac{2\pi}{L} \right)^2$

Density of states 2D

$$E_v = \hbar\omega_c \left(v + \tfrac{1}{2}\right)$$

The number of states between ring v-1 and ring v is

$$\frac{\pi\left(k_v^2 - k_{v-1}^2\right)}{\left(\dfrac{2\pi}{L}\right)^2} \qquad \frac{\hbar^2 k_v^2}{2m} = \hbar\omega_c\left(v + \tfrac{1}{2}\right)$$

$$k_v^2 - k_{v-1}^2 = \frac{2m\omega_c}{\hbar}\left[\left(v + \tfrac{1}{2}\right) - \left(v - 1 + \tfrac{1}{2}\right)\right] = \frac{2m\omega_c}{\hbar}$$

The number of states between ring v-1 and ring v is $\dfrac{m\omega_c}{2\pi\hbar}L^2$

The density of states per spin is $\dfrac{m\omega_c}{2\pi\hbar}$