Quantization

Quantization

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(\vec{r})\psi = E\psi$$

Start with the classical equations of motion
Find the normal modes
Construct the Lagrangian
From the Lagrangian determine the conjugate variables
Perform a Legendre transformation to the Hamiltonian
Quantize the Hamiltonian

Harmonic oscillator

Newton's law: $ma = -Kx$

Euler - Lagrange equations: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$

Lagrangian (constructed by inspection) $L\left(x,\dot{x}\right) = \frac{m\dot{x}^2}{2} - \frac{Kx^2}{2}$

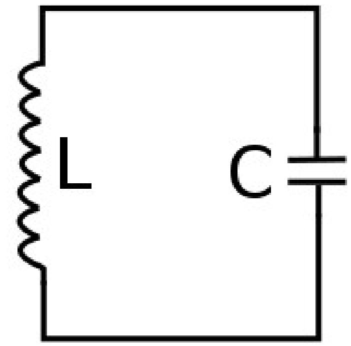
Conjugate variable: $p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$

Legendre transformation: $H = p\dot{x} - L = \frac{p^2}{2m} + \frac{Kx^2}{2}$

Quantize: $p \rightarrow -i\hbar\frac{\partial}{\partial x}$ $H\psi = \frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{Kx^2}{2}\psi$

LC circuit

Classical equations $V = L\frac{dI}{dt}$ $I = -C\frac{dV}{dt}$ $Q = CV$

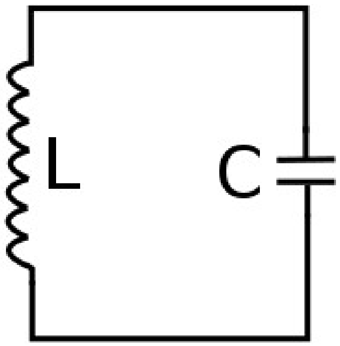


$$\frac{Q}{C} = -L\frac{d^2Q}{dt^2}$$

Euler - Lagrange equation: $$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Q}} - \frac{\partial \mathcal{L}}{\partial Q} = 0$$

Lagrangian (constructed by inspection) $$\mathcal{L}\left(Q,\dot{Q}\right) = \frac{L\dot{Q}^2}{2} - \frac{Q^2}{2C}$$

LC circuit



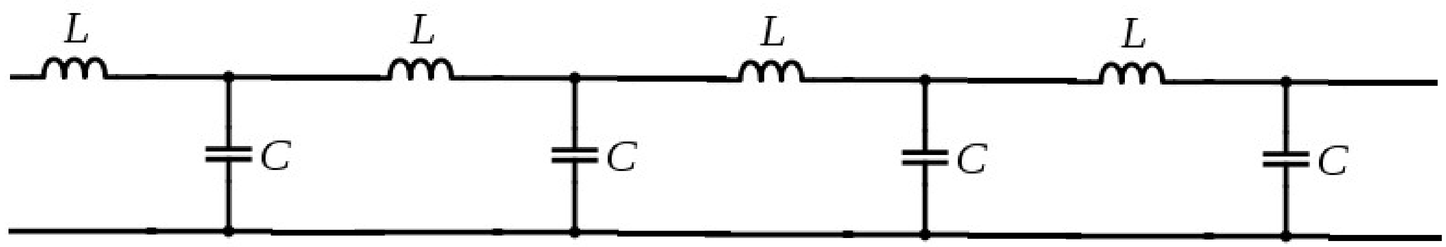
Conjugate variable: $$p = \frac{\partial \mathcal{L}}{\partial \dot{Q}} = L\dot{Q}$$

Legendre transformation: $$H = L\dot{Q}^2 - \mathcal{L} = \frac{L\dot{Q}^2}{2} + \frac{Q^2}{2C}$$

Quantize: $$p \rightarrow -i\hbar \frac{\partial}{\partial Q}$$

$$H\psi = \frac{-\hbar^2}{2L}\frac{d^2\psi}{dQ^2} + \frac{Q^2}{2C}\psi = E\psi$$

Transmission line



L inductance/m
C capacitance/m

$$-\frac{dV}{dx} = L\frac{dI}{dt} \qquad -\frac{dI}{dx} = C\frac{dV}{dt}$$

$$\frac{d^2V}{dx^2} = LC\frac{d^2V}{dt^2}$$

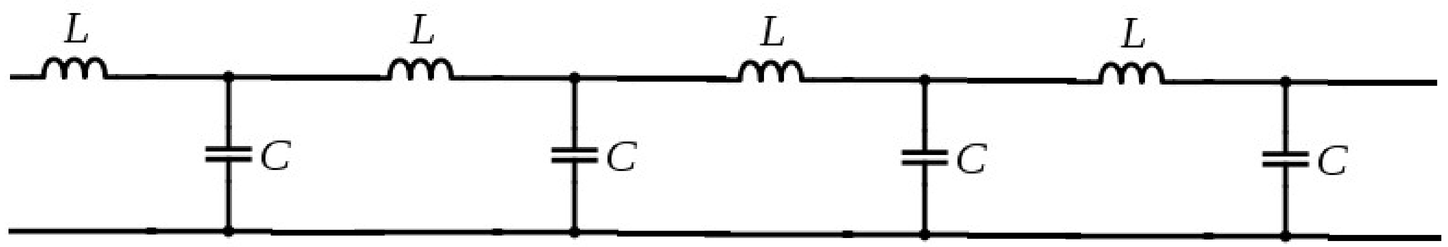
normal mode solution:

$$V_k = V_0 \exp\left(i\left(kx - \omega t\right)\right)$$

$$I_k = I_0 \exp\left(i\left(kx - \omega t\right)\right)$$

Each normal mode moves independently from the other normal modes

Transmission line



Substituting the normal mode solution $V = V_0 \exp\left(i\left(kx - \omega t\right)\right)$

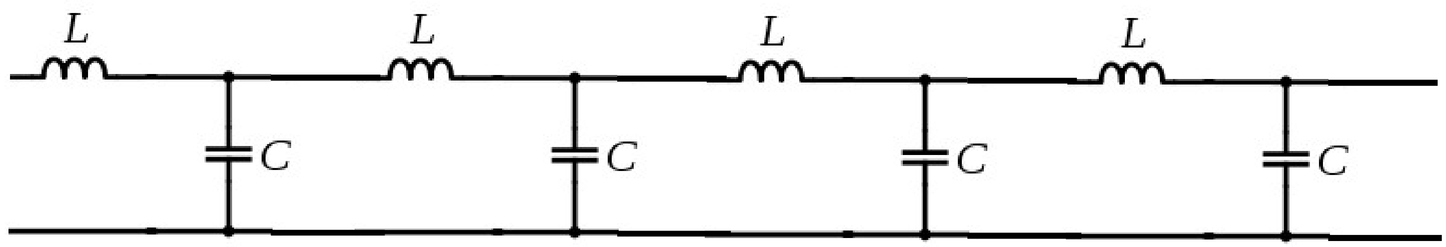
into the wave equation $$\frac{d^2V}{dx^2} = LC\frac{d^2V}{dt^2} \longrightarrow -k^2 = -LC\omega^2$$

yields the dispersion relation $$\omega = \frac{k}{\sqrt{LC}} = ck$$

$$I = \sqrt{\frac{C}{L}}V \qquad Z = \frac{V}{I} = \sqrt{\frac{L}{C}}$$

An infinite transmission line is resistive, typically ~ 50 Ω.

Transmission line



Wave equation

$$c^2 \frac{d^2V}{dx^2} = \frac{d^2V}{dt^2}$$

$$c = \frac{1}{\sqrt{LC}}$$

c is the speed of waves

Not clear what mass we should use in the Schrödinger equation

The Schrödinger equation is for amateurs

Lagrangian (constructed by inspection) $\mathcal{L}(x,\dot{x})$ → Euler - Lagrange equations:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = 0$$

↓

Classical equations of motion (Newton's law)

↓

Conjugate variable: $p = \frac{\partial \mathcal{L}}{\partial \dot{x}}$

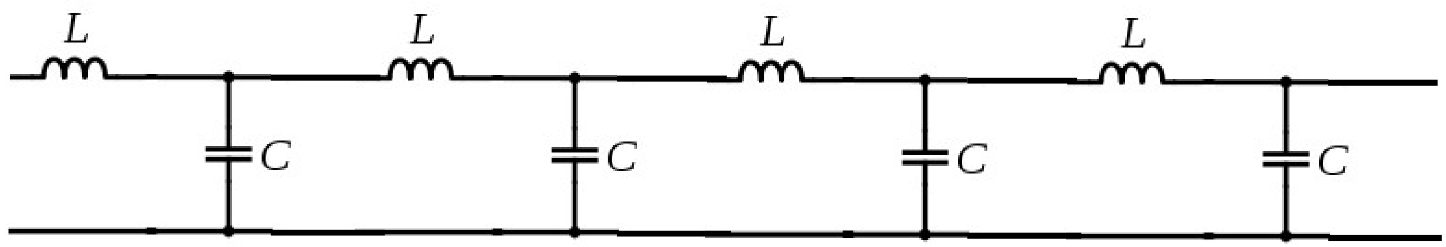
↓

Legendre transformation: $H = p\dot{x} - \mathcal{L}$

↓

Quantize: $p \rightarrow -i\hbar \frac{\partial}{\partial x}$

Transmission line



$$c^2 \frac{d^2V}{dx^2} = \frac{d^2V}{dt^2}$$

normal mode solution: $V_k = V_0 \exp\left(i\left(kx - \omega t\right)\right)$

$$-c^2k^2V_k = \frac{d^2V_k}{dt^2}$$

Each normal mode moves independently from the other normal modes

Lagrangian

Construct the Lagrangian 'by inspection'. The Euler-Lagrange equation and the classical equation of motion for a normal mode are,

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{V}_k}\right) - \frac{\partial \mathcal{L}}{\partial V_k} = 0. \qquad -c^2 k^2 V_k = \frac{\partial^2 V_k}{\partial t^2}.$$

classical equation for the mode k

The Lagrangian is, $$\mathcal{L} = \frac{\dot{V}_k^{\,2}}{2} - \frac{c^2 k^2}{2} V_k^{\,2}$$

Hamiltonian

$$\mathcal{L} = \frac{\dot{V}_k^{\,2}}{2} - \frac{c^2 k^2}{2} V_k^{\,2}$$

The conjugate variable to V_k is,

$$\frac{\partial \mathcal{L}}{\partial \dot{V}_k} = \dot{V}_k$$

The Hamiltonian is constructed by performing a Legendre transformation,

$$H = \dot{V}_k \dot{V}_k - \mathcal{L} = \frac{\dot{V}_k^{\,2}}{2} + \frac{c^2 k^2}{2} V_k^{\,2}$$

To quantize we replace the conjugate variable by $-i\hbar \dfrac{\partial}{\partial V_k}$

$$\frac{-\hbar^2}{2} \frac{d^2\psi}{dV_k^{\,2}} + \frac{c^2 k^2}{2} V_k^{\,2}\psi = E\psi$$

Quantum solutions

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{K}{2}x^2\psi = E\psi \qquad \frac{-\hbar^2}{2}\frac{d^2\psi}{dV_k{}^2}+\frac{c^2k^2}{2}V_k{}^2\psi = E\psi$$

This equation is mathematically equivalent to the harmonic oscillator.

$$E = \hbar\omega\left(j+\tfrac{1}{2}\right) \qquad j = 0, 1, 2, \ldots$$

spring constant →

$$\omega = \sqrt{\frac{K}{m}}$$

mass - spring

$$\omega = \sqrt{c^2k^2}$$

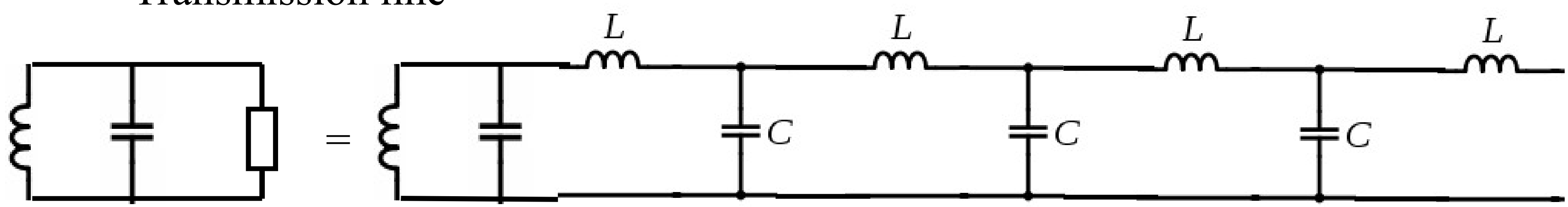
wave mode

$$\omega = c\left|\vec{k}\right|$$

j is the number of photons.

Dissipation in Quantum mechanics

Transmission line

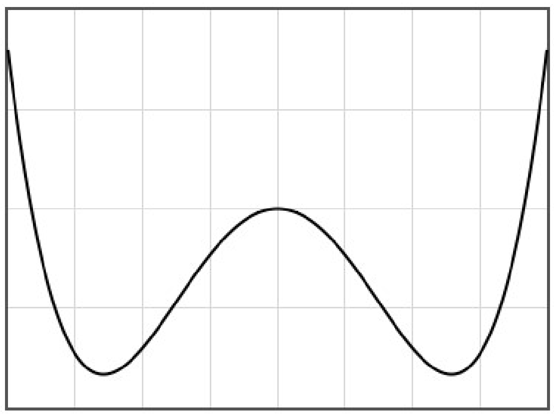


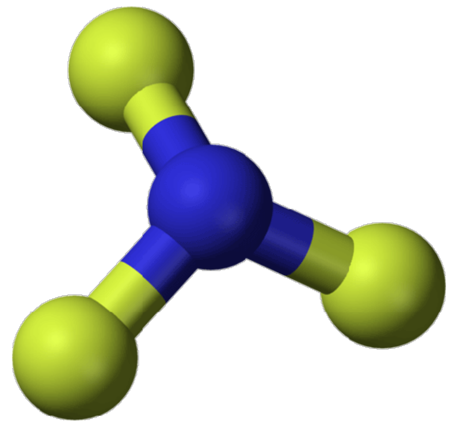
$$I = \sqrt{\frac{C}{L}} V$$

$$Z = \frac{V}{I} = \sqrt{\frac{L}{C}}$$

An infinite transmission line is resistive

Ammonia



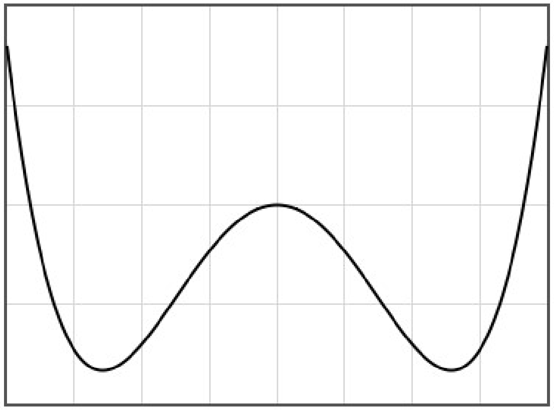


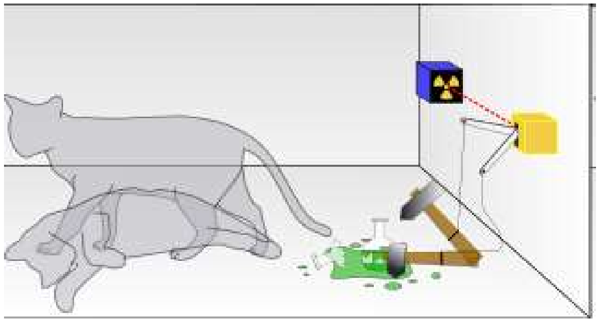
$$\psi_0 \propto \exp\left(-\frac{iE_0 t}{\hbar}\right)$$

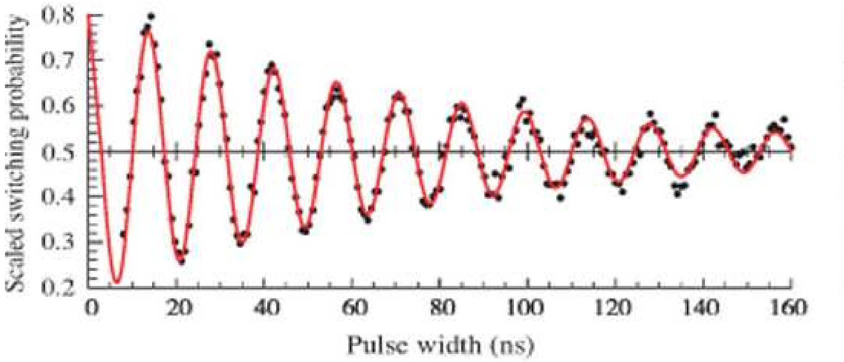
$$\psi_1 \propto \exp\left(-\frac{iE_1 t}{\hbar}\right)$$

Dissipation in quantum mechanics

Quantum coherence is maintained until the decoherence time. This depends on the strength of the coupling of the quantum system to other







Decay is the decoherence time.

Dissipation in solids

At zero electric field, the electron eigen states are Bloch states. Each Bloch state has a k vector. The average value of $k = 0$ (no current).

At finite electric field, the Bloch states are no longer eigen states but we can calculate the transitions between Bloch states using Fermi's golden rule. The final state may include an electron state plus a phonon. The average value of k is not zero (finite current).

The phonons carry the energy away like a transmission line.

The quantization of the electromagnetic field

Wave nature and the particle nature of light

Unification of the laws for electricity and magnetism (described by Maxwell's equations) and light

Quantization of fields

Derive the Bose-Einstein function

Planck's radiation law

Serves as a template for the quantization of noninteracting bosons: phonons, magnons, plasmons, and other quantum particles that inhabit solids.

http://lamp.tu-graz.ac.at/~hadley/ss2/emfield/quantization_em.php