15. Linear Response Theory

Nov. 22, 2018

Fluctuation-dissipation theorem

Brownian motion: The response to thermal noise is related to the viscosity.

$$m\frac{dv}{dt} = -\mu v \qquad\qquad D = \mu k_B T$$

Johnson noise: The voltage fluctuations are related to the resistance.

$$V_{rms} = \sqrt{4k_B TRB}$$

The fluctuation-dissipation theorem holds at equilibrium (where the equations are linear to a good approximation).

http://en.wikipedia.org/wiki/Fluctuation_dissipation_theorem

Dielectric response of insulators

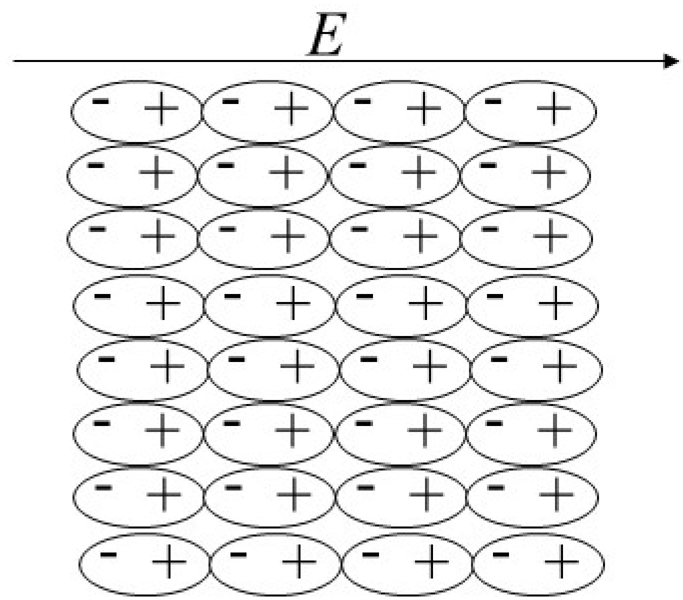
The electric polarization is related to the electric field

$$P_i = \varepsilon_0 \chi_{ij} E_j$$

The electric displacement vector *D* is also related to the electric field

$$D_i = P_i + \varepsilon_0 E_i = \varepsilon_0 (1 + \chi_{ij}) E_j = \varepsilon_0 \varepsilon_{ij} E_j$$

$$\varepsilon_{ij} = (1 + \chi_{ij})$$



E is decreased by a factor of the dielectric constant

Dielectric response of insulators

In an insulator, charge is bound. The response to an electric field can be modeled as a collection of damped harmonic oscillators

$$P = nex$$

Macroscopic polarization

density

ex = dipole moment

The core electrons of a metal respond to an electric field like this too.

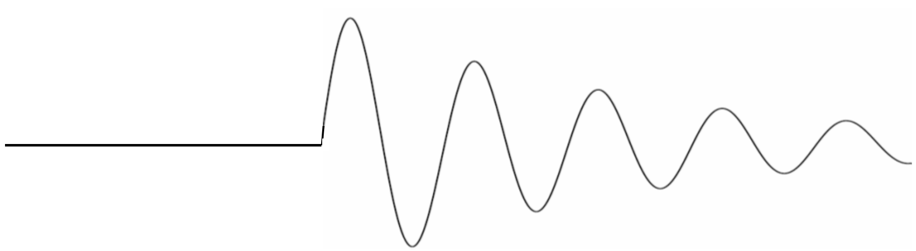
Dielectric response of insulators

The differential equation that describes how the position of the charge changes in time is:

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=-eE(t)$$

The impulse response function is:

$$g(t)=-\frac{1}{b}\exp\left(\frac{-bt}{2m}\right)\sin\left(\frac{\sqrt{4mk-b^2}}{2m}t\right)\qquad t>0$$



Electric susceptibility

$$\vec{P} = \varepsilon_0 \chi_E \vec{E} \qquad \vec{P} = nq\vec{x}$$

$$\chi_E = \frac{P}{\varepsilon_0 E} = \frac{nqx}{\varepsilon_0 E}$$

response/drive

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = qE(t)$$

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = -\frac{qE}{m}$$

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \gamma = \frac{b}{m} = \frac{1}{\tau}$$

Electric susceptibility

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = -\frac{qE}{m}$$

Assume a solution of the form $x(\omega)e^{i\omega t}$, $E(\omega)e^{i\omega t}$

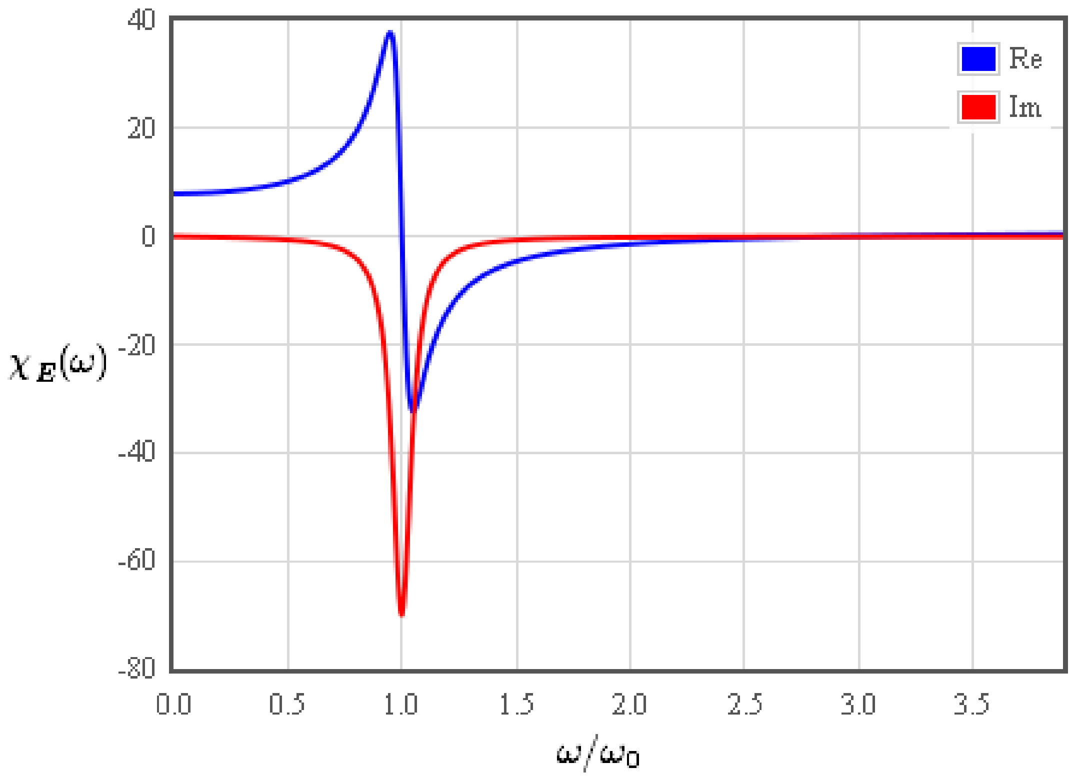
$$-\omega^2 x(\omega) + i\omega\gamma x(\omega) + \omega_0^2 x(\omega) = qE(\omega)$$

$$\frac{x(\omega)}{qE(\omega)} = \frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

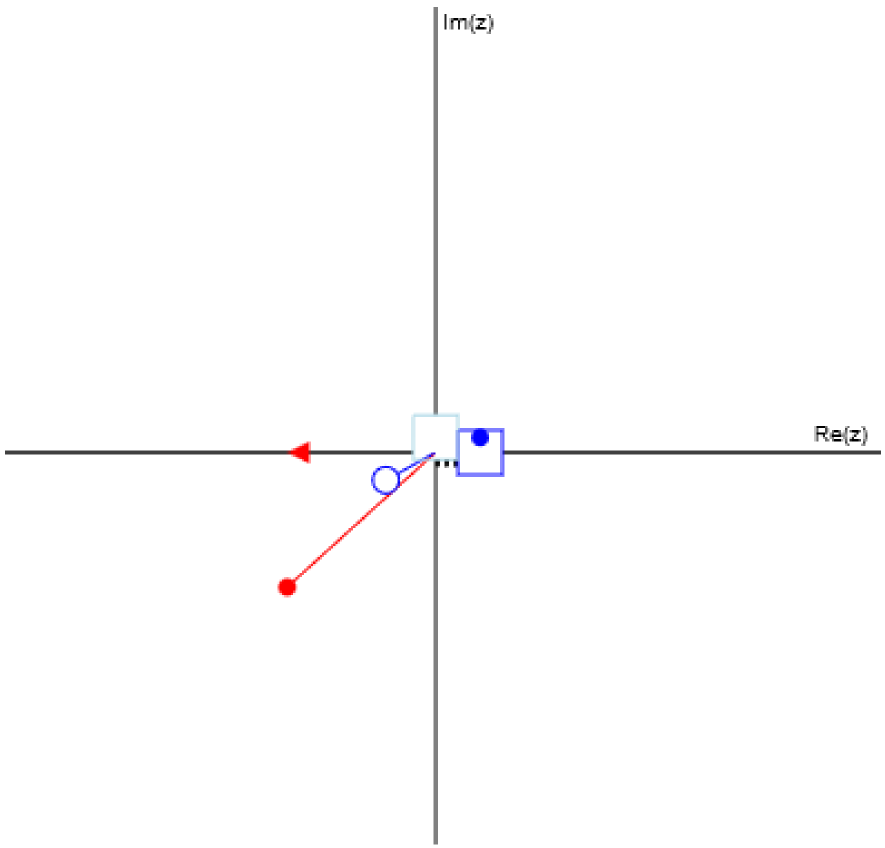
$$\chi_E(\omega) = \frac{n_{\omega_0} q^2}{\varepsilon_0 m} \frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

Electric susceptibility

$$\chi_E(\omega) = \frac{n_{\omega_0} q^2}{\varepsilon_0 m} \frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega}$$



Resonance of a damped driven harmonic oscillator



m =4 [kg]

b =1 [N s/m]

k =6 [N/m]

F_0 =0.9 [N]

ω =0.8 [rad/s]

$\omega_0 = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$ = 1.22 [rad/s] = 0.194 [Hz]

$\theta = \text{atan}\left(\frac{\omega b}{k - m\omega^2}\right)$ = 0.228 [rad] = 13.1 [deg]

$|A| = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \omega^2 b^2}}$ = 0.255 [m]

$Q = \frac{\sqrt{mk}}{b}$ = 4.90

Display $F_0 e^{i\omega t}$: ☑ Display $|A|e^{i(\omega t - \theta)}$: ☑

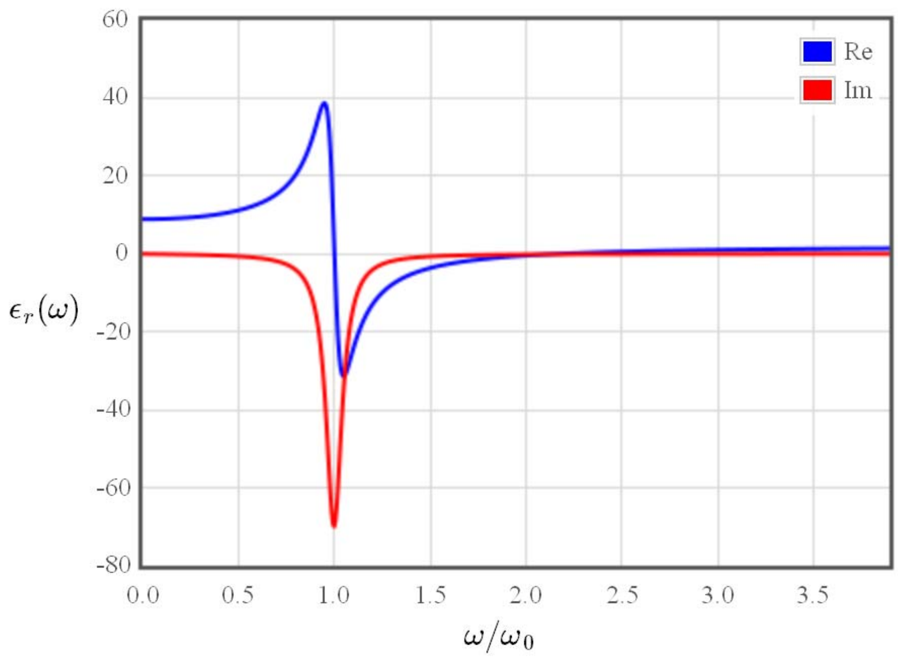
Display transients z: ☑ Display x_2: ☑

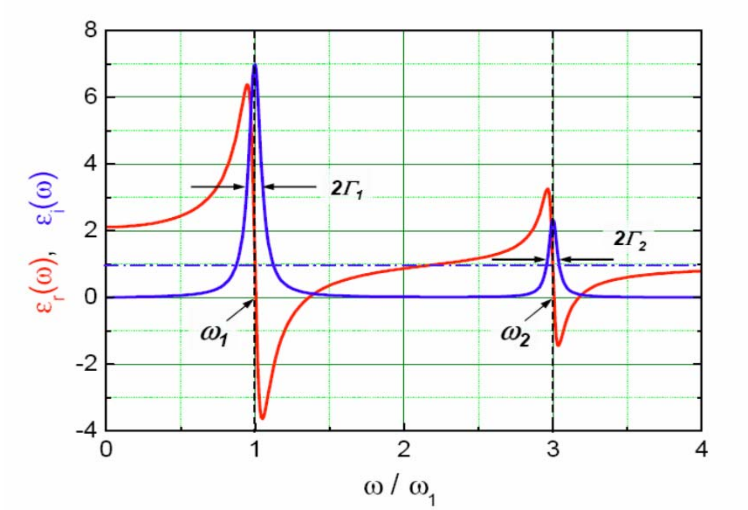
http://lamp.tu-graz.ac.at/~hadley/physikm/apps/resonance.en.php

Dielectric function

$$\vec{D} = \epsilon_r \epsilon_0 \vec{E} = \vec{P} + \epsilon_0 \vec{E}$$

$$\varepsilon_r(\omega) = 1 + \chi_E(\omega) = \varepsilon_\infty + \frac{n_{\omega_0} q^2}{\varepsilon_0 m} \frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

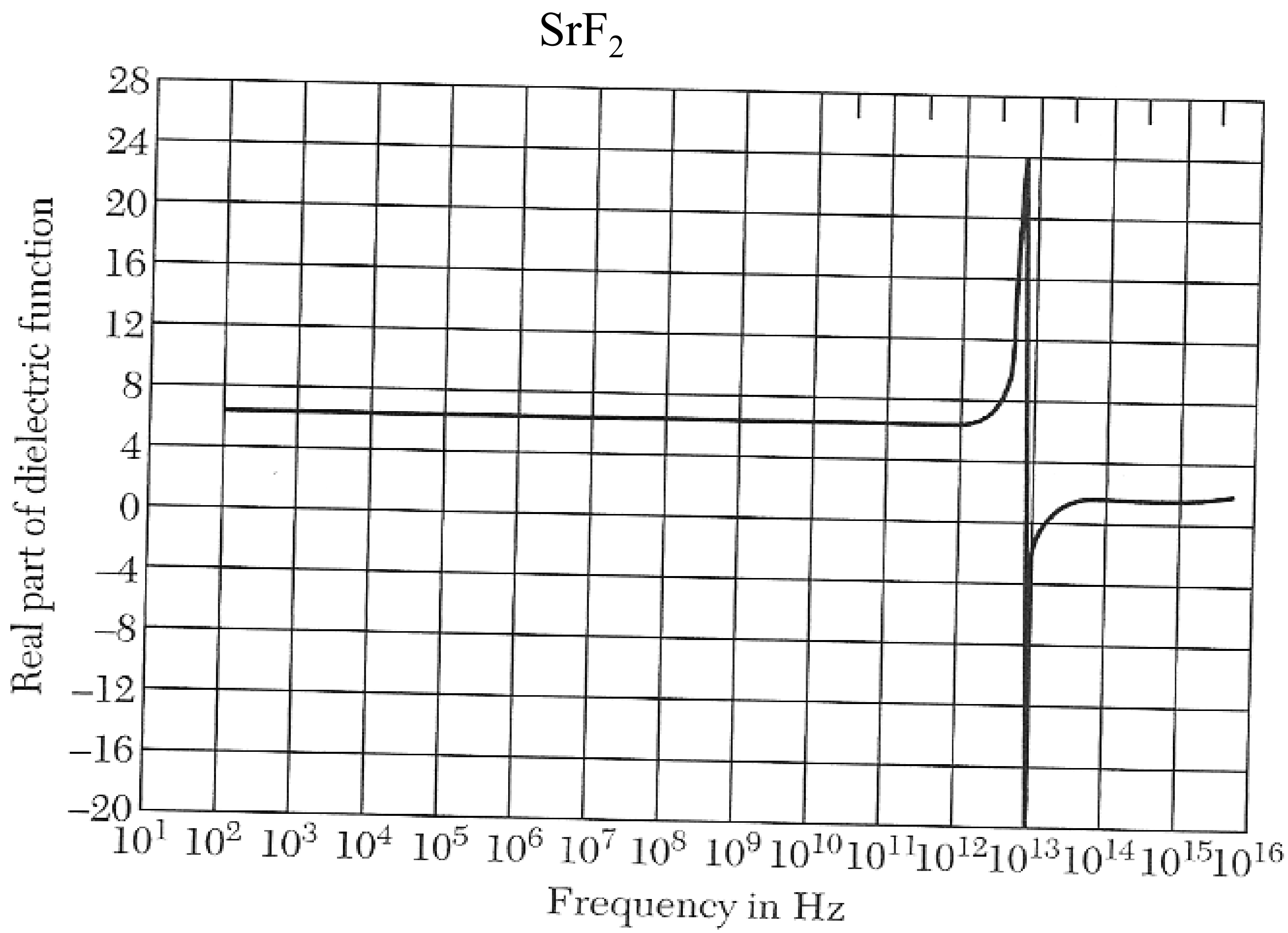




Gross and Marx

There can be more resonances.

Dielectric function of insulators



Insulators can often be modeled as a simple resonance.

Dispersion relation

Maxwell equations in matter $\Rightarrow$ Wave equation.

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

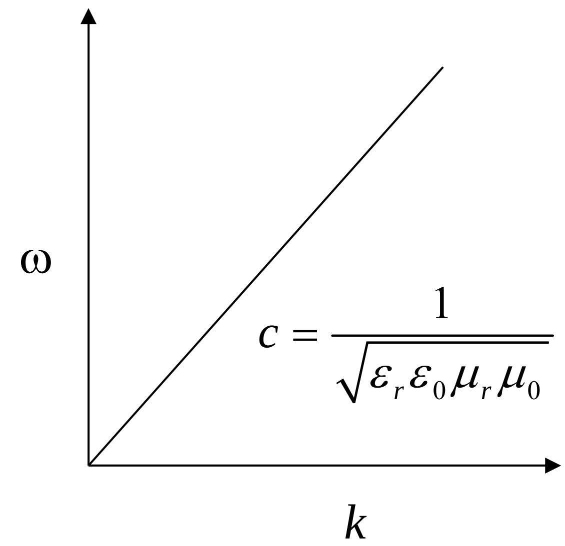
$$\vec{D} = \varepsilon \vec{E} \qquad \vec{B} = \mu \vec{H}$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

Take the curl $$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial \nabla \times \vec{H}}{\partial t}$$

$$\nabla\left(\overset{0}{\cancel{\nabla \cdot \vec{E}}}\right) - \nabla^2 \vec{E} = -\mu \frac{\partial^2 \vec{D}}{\partial t^2}$$

$$\varepsilon_r \varepsilon_0 \mu_r \mu_0 \frac{\partial^2 \vec{D}}{\partial t^2} = \nabla^2 \vec{D}$$



The normal mode solutions are plane waves: $\vec{D} = \vec{D}_0 \exp\left(\mathrm{i}\left(\vec{k} \cdot \vec{r} - \omega t\right)\right)$

$$\boxed{\varepsilon(\omega, k)\mu_0 \varepsilon_0 \omega^2 = k^2}$$

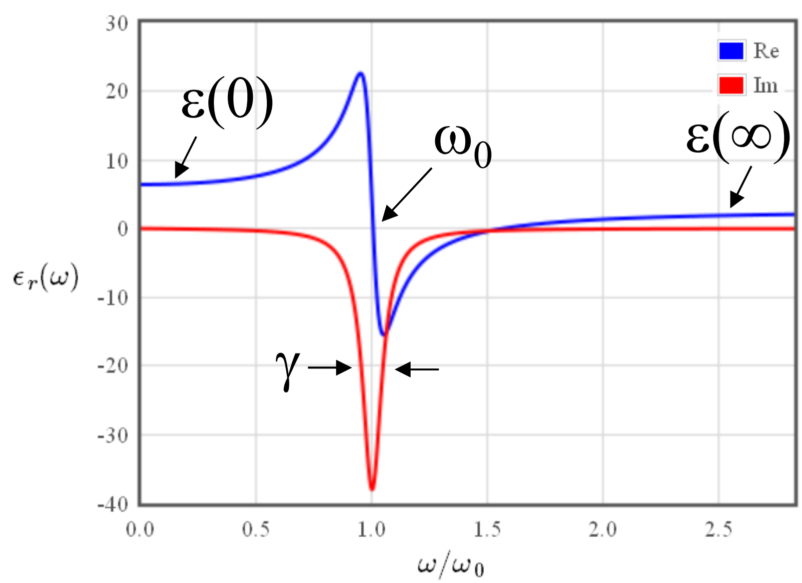
Dispersion relation

$$\varepsilon(\omega)\mu_0\varepsilon_0\omega^2 = k^2$$

If ε is real and positive: propagating electromagnetic waves $\exp\left(i\left(\vec{k}\cdot\vec{r}-\omega t\right)\right)$

If $\varepsilon_r < 0$: decaying solutions $\exp(-\vec{k}\cdot\vec{r}-i\omega t)$

If ε is complex, $\varepsilon_r > 0$: decaying electromagnetic waves $\exp\left(i\left(\vec{k}\cdot\vec{r}-\omega t\right)\right)\exp(-\kappa r)$



Dielectric function

Dispersion relation: $\varepsilon_r \mu_0 \varepsilon_0 \omega^2 = k^2$

$$k = \sqrt{\varepsilon_r \mu_0 \varepsilon_0}\,\omega = \frac{\sqrt{\varepsilon_r}\,\omega}{c}$$

Measurable: $\sqrt{\varepsilon} = n + iK$

refractive index (n) — extinction coefficient (K)

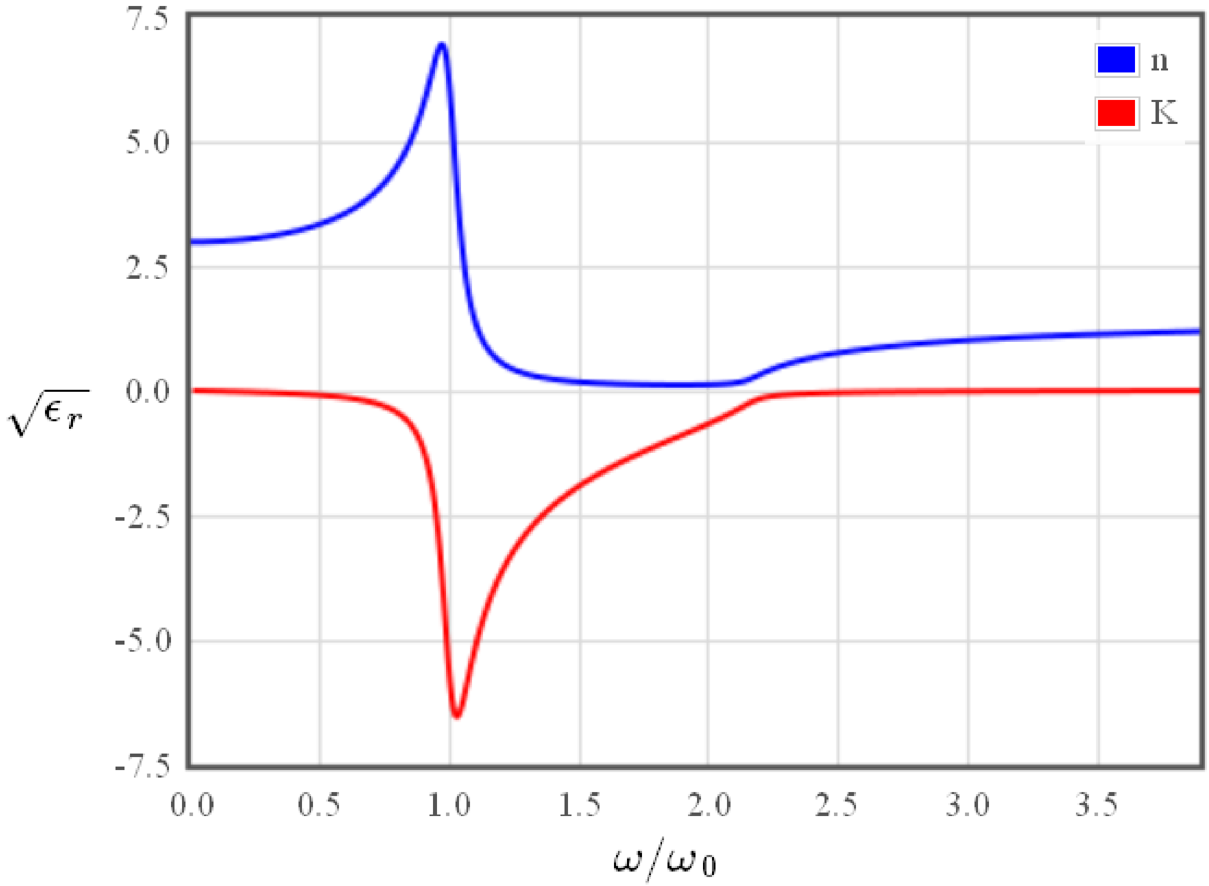
Vacuum	Dielectric
$\omega = ck$	$k = \frac{\sqrt{\varepsilon_r}\,\omega}{c}$
$\exp\left(i\left(\frac{\omega x}{c} - \omega t\right)\right)$	$\exp\left(i\left(\frac{(n+iK)\omega x}{c} - \omega t\right)\right)$
	$= \exp\left(i\left(\frac{n\omega x}{c} - \omega t\right)\right)\exp\left(-\frac{\omega K x}{c}\right)$

Intensity $I(x) = I(0)\exp(-\alpha x)$ J m^{-2} s^{-1} Beer-Lambert

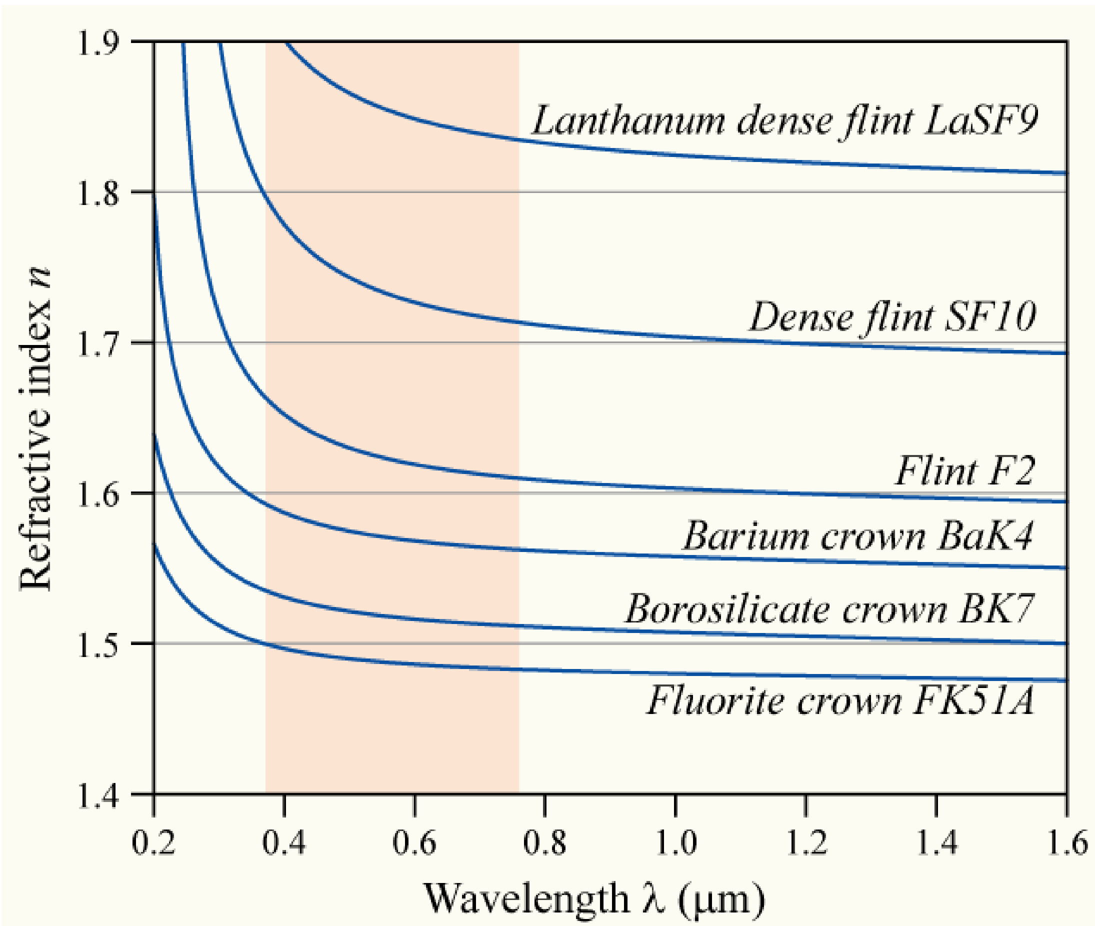
absorption coefficient $\longrightarrow$ $\alpha = \frac{2\omega K}{c}$

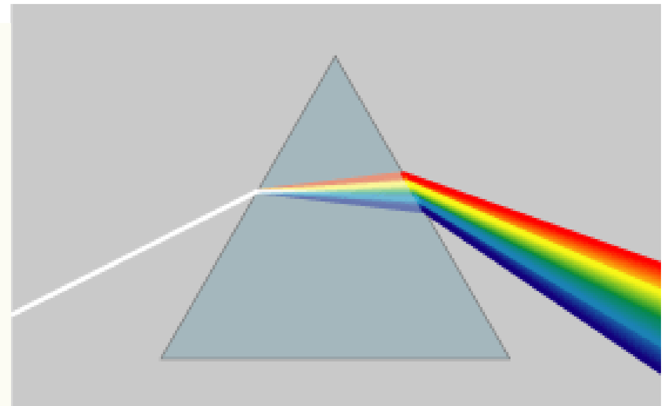
The index of refraction *n* and the extinction coefficient *K*

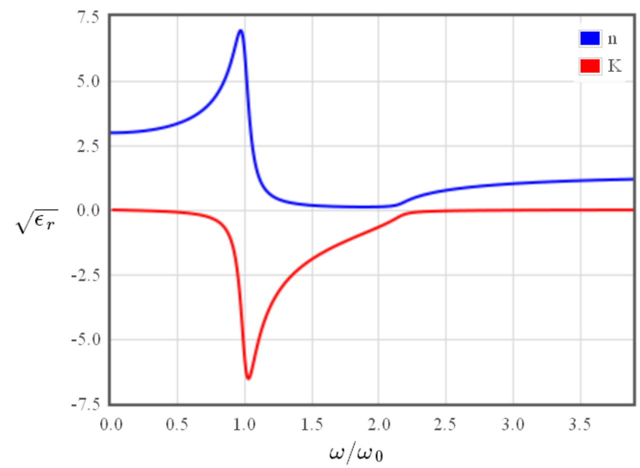
$$\sqrt{\epsilon_r} = n + iK$$



Dispersion







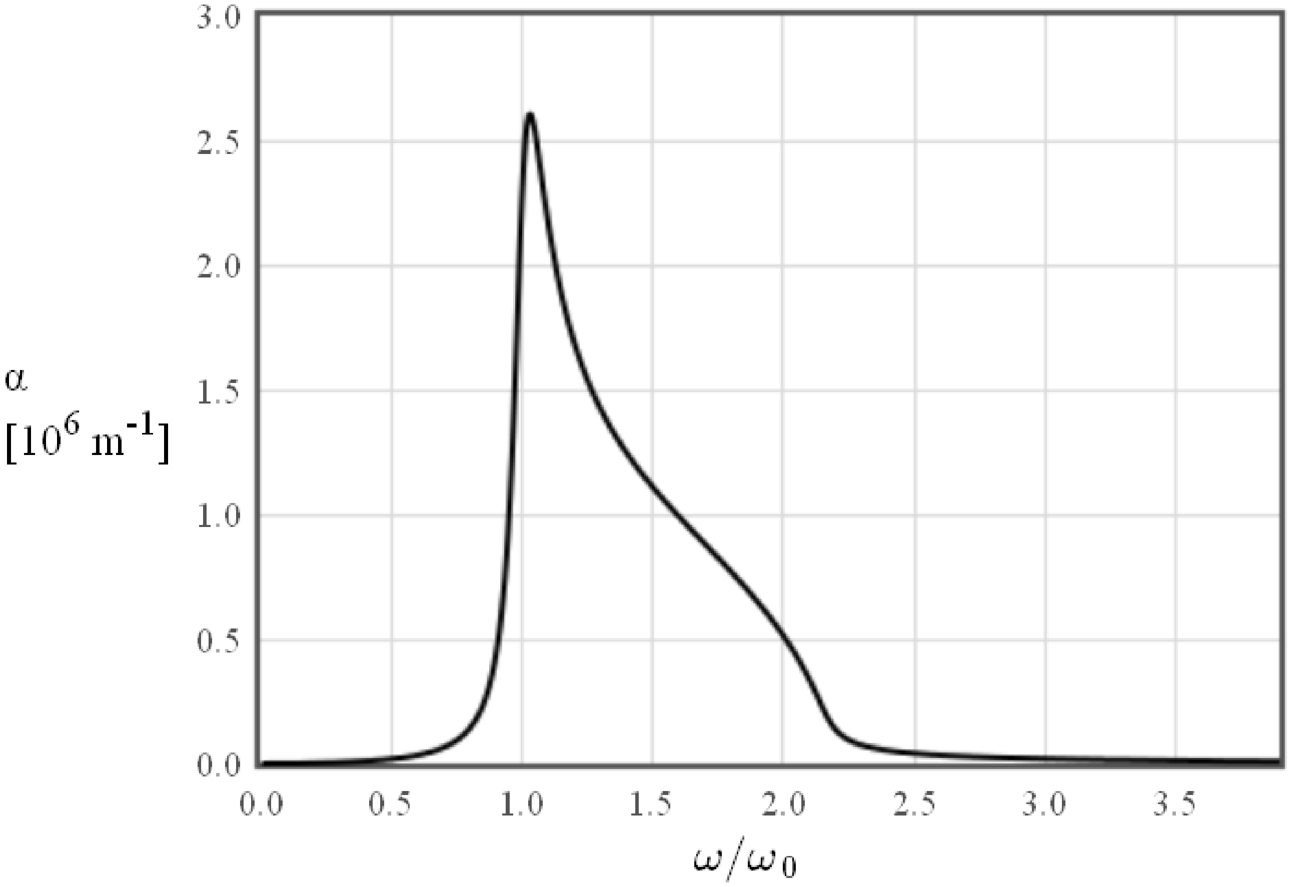
Cause of chromatic aberration in lenses.

http://en.wikipedia.org/wiki/Dispersion_%28optics%29#mediaviewer/File:Prism_rainbow_schema.png
http://en.wikipedia.org/wiki/Refractive_index

Absorption coefficient α

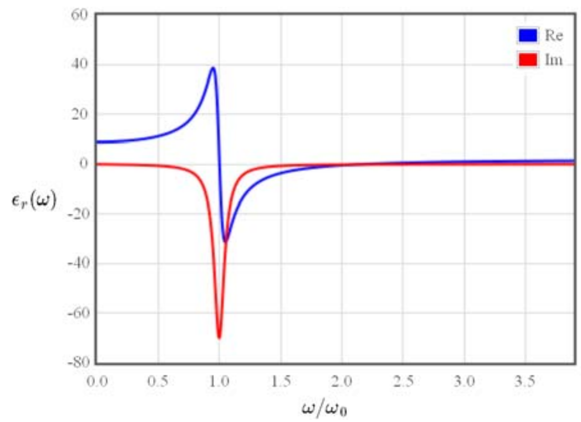
$$I = I_0 \exp(-\alpha x)$$

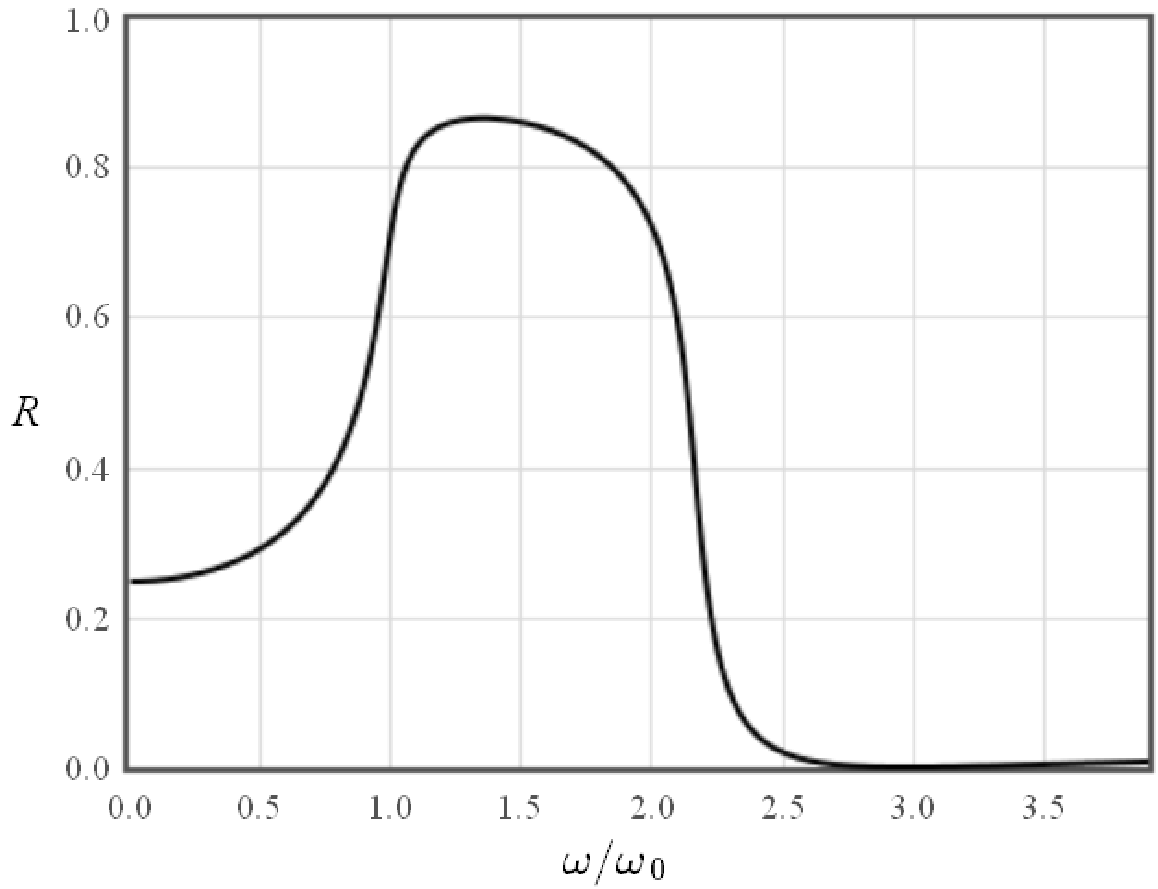
$$\alpha = \frac{2\omega K}{c}$$



Reflectance

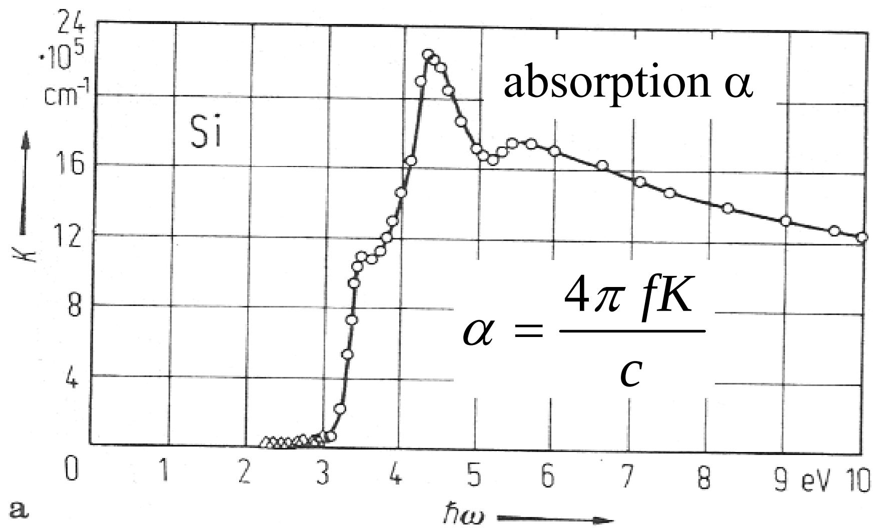
$$R = \frac{(n-1)^2 + K^2}{(n+1)^2 + K^2}$$



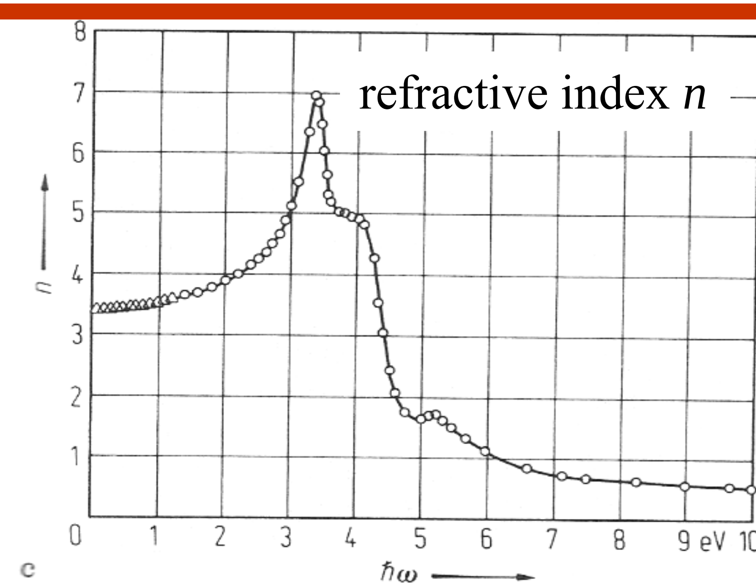


Dielectric function of silicon

$$\sqrt{\varepsilon(\omega)} = n(\omega) + iK(\omega)$$



a



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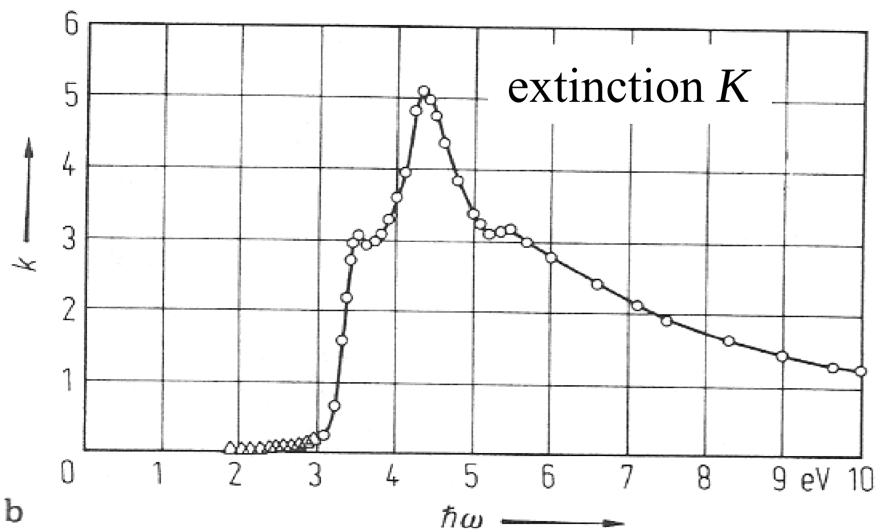

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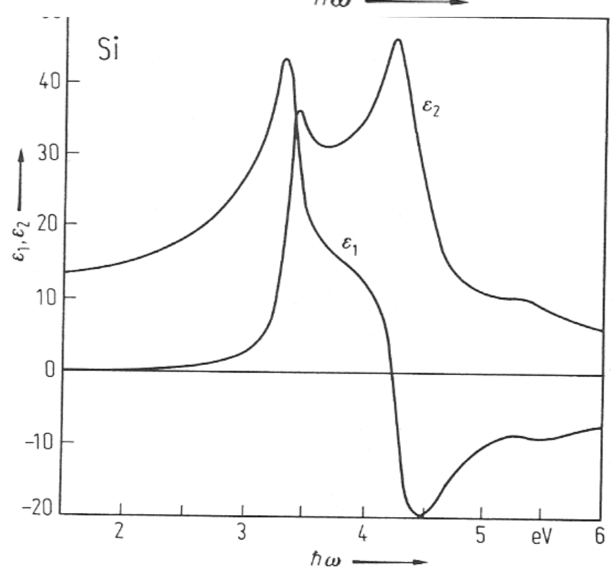

Optical properties of insulators and semiconductors

In an insulator, all charges are bound. By applying an electric field, the electrons and ions can be pulled out of their equilibrium positions. When this electric field is turned off, the charges oscillate as they return to their equilibrium positions. A simple model for an insulator can be constructed by describing the motion of the charge as a damped mass-spring system. The differential equation that describes the motion of a charge is,

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=-qE.$$

Rewriting above equation using $\omega_0=\sqrt{\frac{k}{m}}$ and the damping constant $\gamma=\frac{b}{m}$ yields,

$$\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+\omega_0^2x=-\frac{qE}{m}.$$

If the electric field is pulsed on, the response of the charges is described by the **impulse response function** $g(t)$. The impulse response function satisfies the equation,

$$\frac{d^2g}{dt^2}+\gamma\frac{dg}{dt}+\omega_0^2g=-\frac{q}{m}\delta(t).$$

The solution to this equation is zero before the electric field is pulsed on and at the time of the pulse the charges suddenly start oscillating with the frequency $\omega_1=\sqrt{\omega_0^2-\frac{\gamma^2}{4}}$. The amplitude of the oscillation decays exponentially to zero in a characteristic time $\frac{2}{\gamma}$.

$$g(t)=-\frac{q}{m\omega_1}\exp\left(-\frac{\gamma}{2}t\right)\sin(\omega_1 t).$$