17. Superconductivity / Linear Response

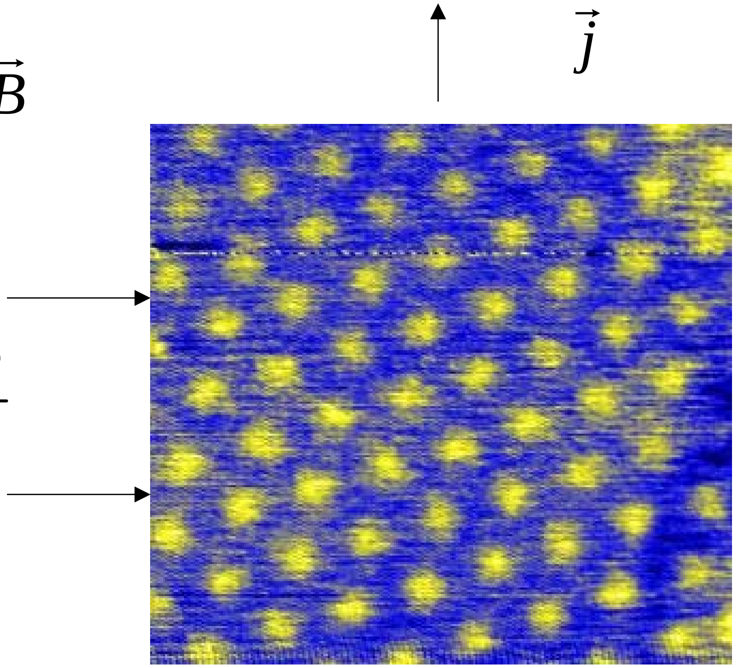
Dec. 2, 2019

Vortices in Superconductors

Lorentz force $\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$

$$\vec{j} = nq\vec{v}$$

$$\vec{F} = \frac{1}{n}\vec{j} \times \vec{B}$$



Faraday's law $V = -\frac{d\Phi}{dt}$

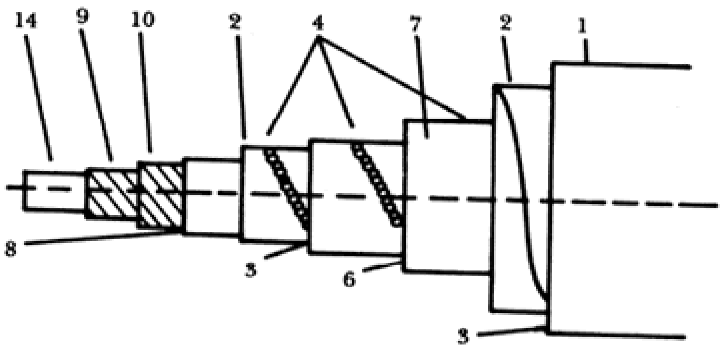
Defects are used to pin the vortices

Superconducting Magnets

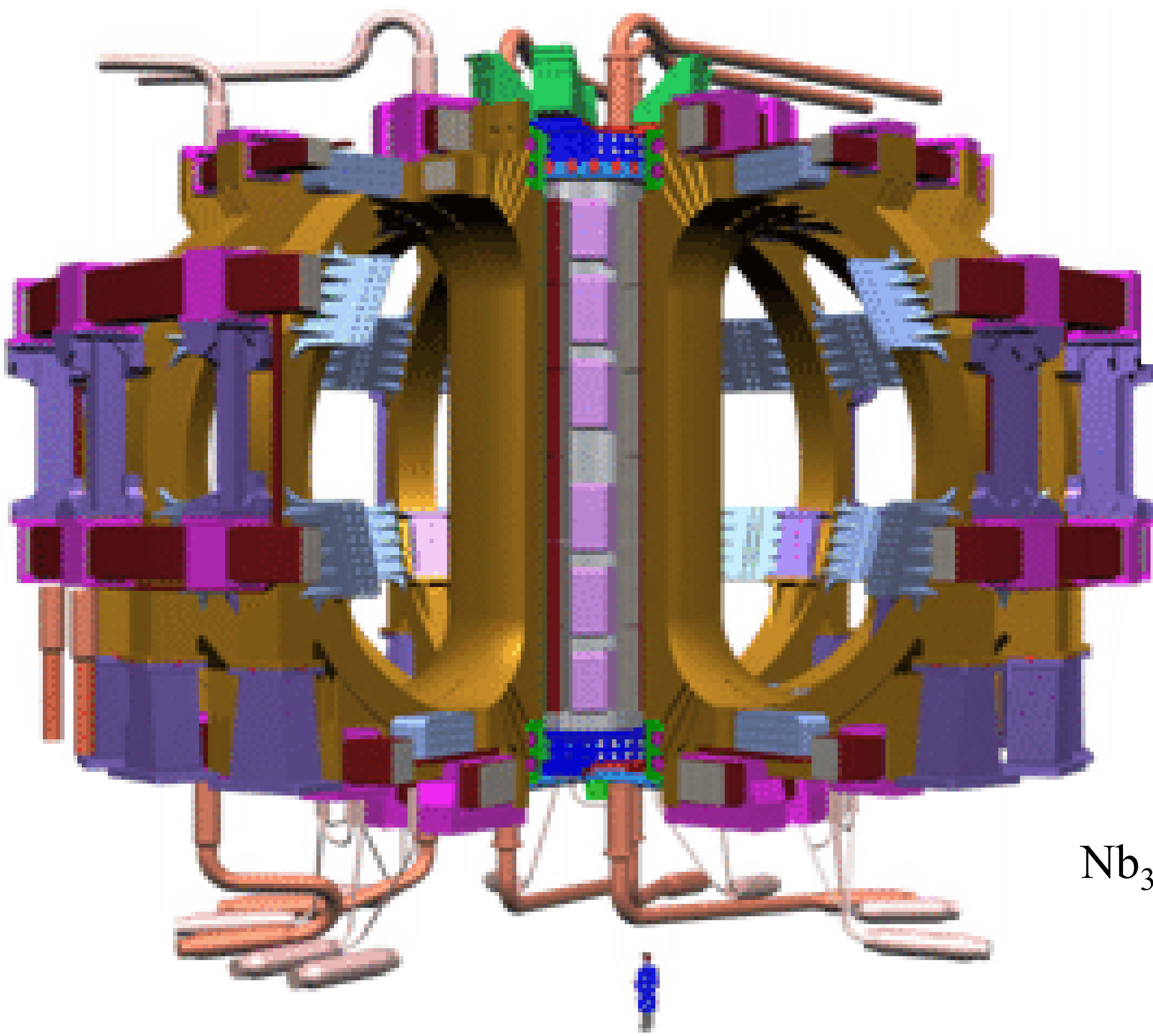
Whole body MRI

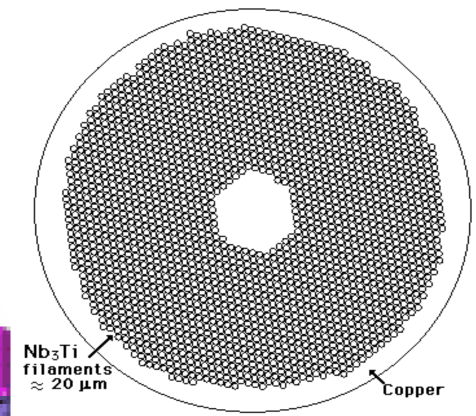
Magnets and cables

Maglev trains



ITER

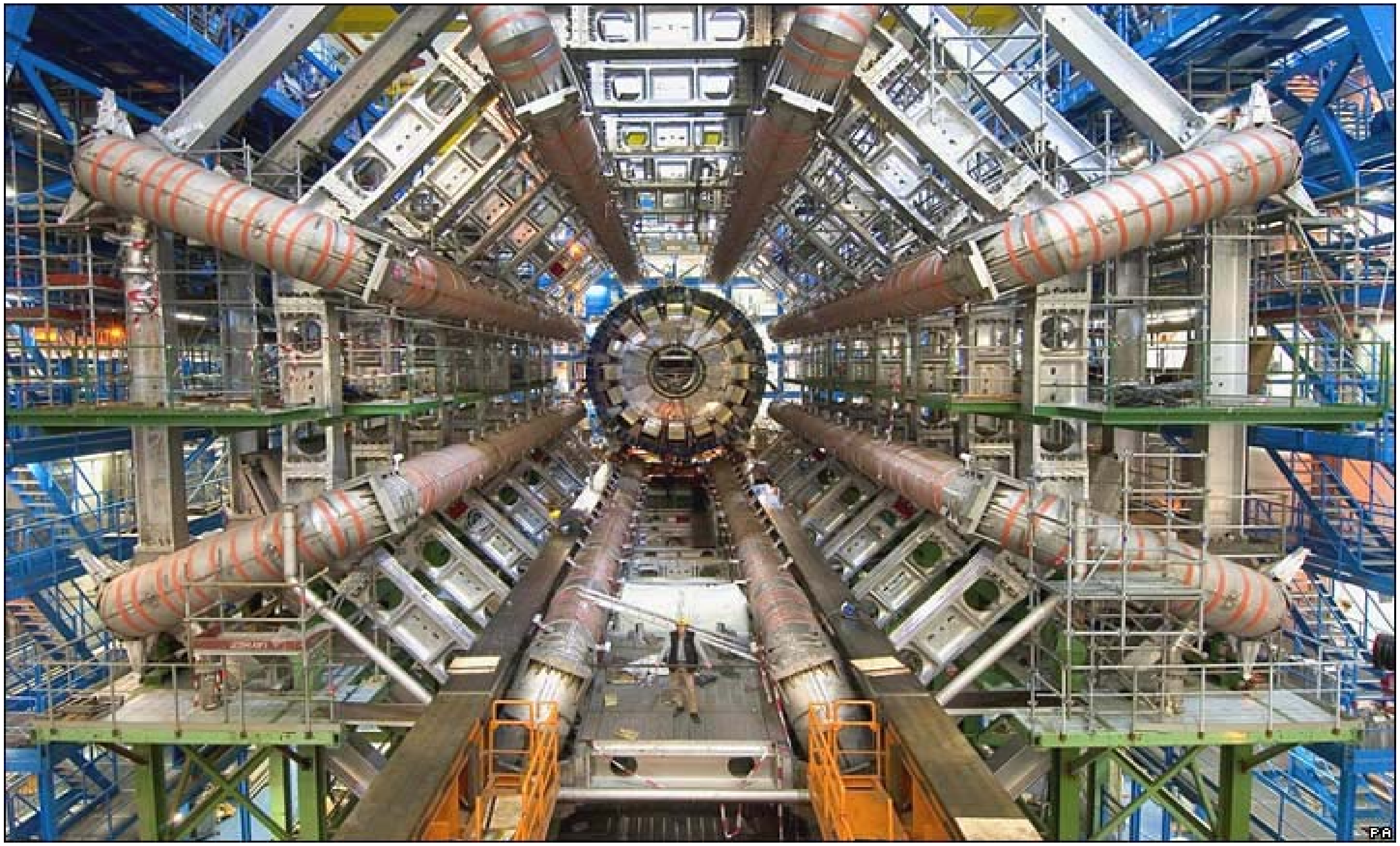




Magnet wire

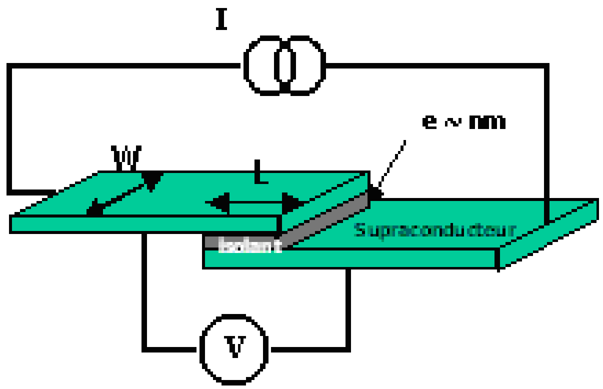
Nb_3Sn Magnet

Superconducting magnets

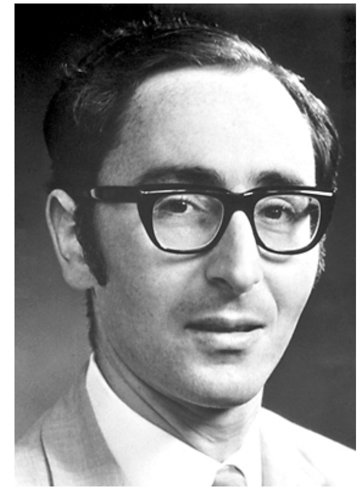


Largest superconducting magnet, CERN
21000 Amps

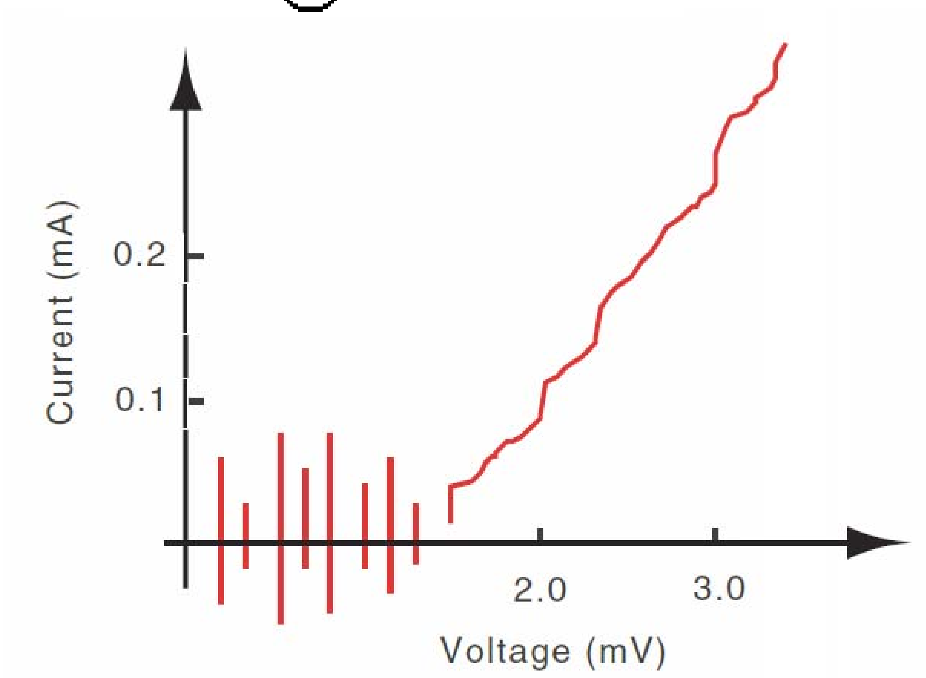
ac - Josephson effect



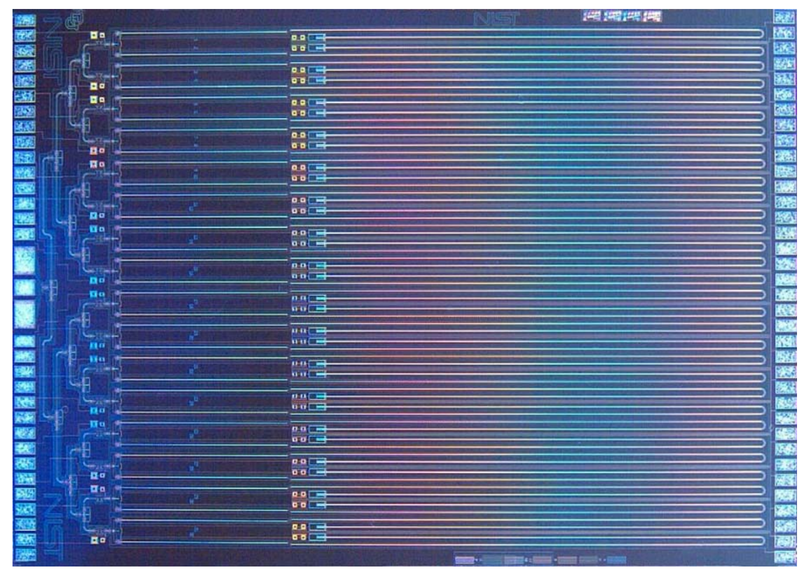
$$V = -\frac{d\Phi}{dt} = n\Phi_0 f = \frac{nhf}{2e}$$



Brian Josephson



DOI: 10.1140/epjst/e2009-01050-6



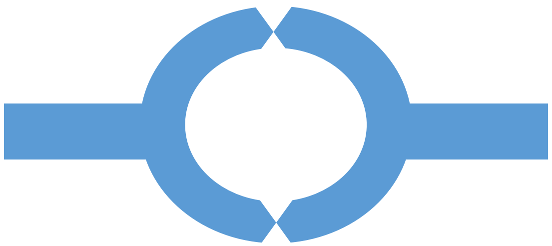
10 V standard

http://www.nist.gov/pml/history-volt/superconductivity_2000s.cfm

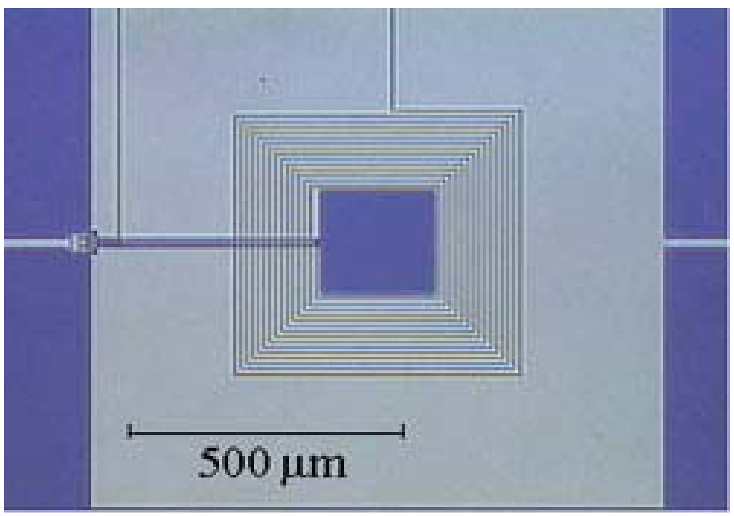
http://www.lne.eu/en/r_and_d/electrical_metrology/josephson-effect-ej.asp

SQUID

Superconducting quantum interference device



$10^{-6}\ \Phi_0\ /\ (Hz)^{1/2}$



Sensitive detectors

10^{-20} m/ $(Hz)^{1/2}$

Gravity wave detector

Linear Response Theory

Classical linear response theory

Fourier transforms
Impulse response functions (Green's functions)
Generalized susceptibility
Causality
Kramers-Kronig relations
Fluctuation - dissipation theorem
Dielectric function
Optical properties of solids

Print version

Numerical Methods

Fourier analysis of real data sets

Consider a series of N measurements x_n that are made at equally spaced time intervals Δt. The total time to make the measurement series is $N\Delta t$. A discrete Fourier transform can be used to find a periodic function $x(t)$ with a fundamental period $N\Delta t$ that passes through all of the points. This function can be expressed as a Fourier series in terms of sines and cosines,

$$x(t) = \sum_{n=0}^{n<N/2} \left[a_n \cos\left(\frac{2\pi nt}{N\Delta t}\right) + b_n \sin\left(\frac{2\pi nt}{N\Delta t}\right)\right].$$

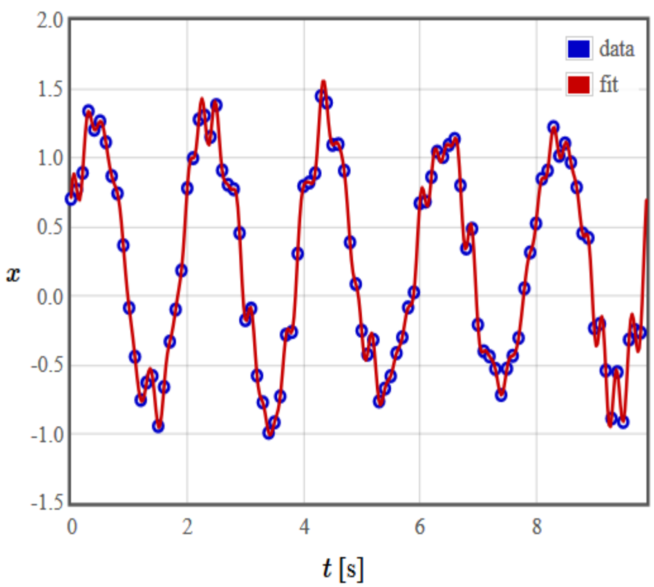
Data for x_n can be input in the textbox below. When the 'Calculate Fourier Coefficients' button is pressed, the periodic function $x(t)$ is plotted through the data points. The Fourier coefficients are tabulated and plotted as well. The fft algorithm first checks if the number of data points is a power-of-two. If so, it calculates the discrete Fourier transform using a Cooley-Tukey decimation-in-time radix-2 algorithm. If the number of data points is not a power-of-two, it uses Bluestein's chirp z-transform algorithm. The fft code was taken from Project Nayuki.

x_n

0.704755992151468
0.7702905111005827
0.8931618373710344
1.3406823044010674
1.2059826464418861
1.2675358230469096
1.1156175628382647
0.8703050439010842
0.7442227455673327
0.3681609224807739
-0.08539320011647894

$\Delta t =$ 0.1 s

Calculate Fourier Coefficients



http://lampx.tugraz.at/~hadley/num/ch3/3.3a.php

Notations for Fourier Transforms

$$F_{-1,-1}\left(\vec{k}\right) = \frac{1}{\left(2\pi\right)^{d}} \int f\left(\vec{r}\right) e^{-i\vec{k}\cdot\vec{r}}\, d\vec{r}\,.$$

$$f\left(\vec{r}\right) = \int F_{-1,-1}\left(\vec{k}\right) e^{i\vec{k}\cdot\vec{r}}\, d\vec{k}\,.$$

$f(r)$ is built of plane waves

$\exp(-\lvert a\rvert x)$	$\frac{\lvert a\rvert}{\pi(a^2+k^2)}$	$\frac{2\lvert a\rvert}{a^2+k^2}$
$\operatorname{sgn}(x)$ $\operatorname{sgn}(x)=-1$ for $x<0$ and $\operatorname{sgn}(x)=1$ for $x>0$	$\frac{-i}{\pi\omega}$	$\frac{-2i}{\omega}$
$\operatorname{sgn}(x)\exp(-\lvert a\rvert x)$	$\frac{-ik}{\pi(a^2+k^2)}$	$\frac{-i2k}{a^2+k^2}$
$H(x)\exp(-\lvert a\rvert x)$	$\frac{\lvert a\rvert-ik}{2\pi(a^2+k^2)}$	$\frac{\lvert a\rvert-ik}{a^2+k^2}$
$\sqcap(x)=H\left(x+\frac{1}{2}\right)H\left(\frac{1}{2}-x\right)$ Square pulse: height = 1, width = 1, centered at $x=0$.	$\frac{\sin(k/2)}{\pi k}$	$\frac{2\sin(k/2)}{k}$
$\sqcap\left(\frac{x-x_0}{a}\right)$ Square pulse: height = 1, width = a, centered at x_0.	$\frac{\sin(ka/2)}{\pi k}\exp(-ikx_0)$	$\frac{2\sin(ka/2)}{k}\exp(-ikx_0)$
$\exp\left(i\vec{k}_0\cdot\vec{r}\right)$ Plane wave	$\delta\left(\vec{k}-\vec{k}_0\right)$	$(2\pi)^d\delta\left(\vec{k}-\vec{k}_0\right)$
1	$\delta(k)$	$2\pi\delta(k)$
$\delta(x)$ $\delta\left(\frac{\vec{r}-\vec{r}_0}{a}\right)$	$\frac{1}{2\pi}$ $\left(\frac{a}{2\pi}\right)^d\exp\left(-i\vec{k}\cdot\vec{r}_0\right)$	1 $a^d\exp\left(-i\vec{k}\cdot\vec{r}_0\right)$
$\exp\left(-\frac{\lvert\vec{r}-\vec{r}_0\rvert^2}{a^2}\right)$	$\left(\frac{a}{2\sqrt{\pi}}\right)^d\exp\left(-\frac{a^2k^2}{4}\right)\exp\left(-i\vec{k}\cdot\vec{r}_0\right)$	$(a\sqrt{\pi})^d\exp\left(-\frac{a^2k^2}{4}\right)\exp\left(-i\vec{k}\cdot\vec{r}_0\right)$
$H(R-\lvert\vec{r}-\vec{r}_0\rvert)$ Disc of radius R centered at $\vec{r}_0$, $\vec{r}\in\mathrm{R}^2$ $H(R-\lvert\vec{r}-\vec{r}_0\rvert)$ Sphere of radius R centered at $\vec{r}_0$, $\vec{r}\in\mathrm{R}^3$	$\frac{R}{2\pi\lvert\vec{k}\rvert}J_1(\lvert\vec{k}\rvert R)\exp(-i\vec{k}\cdot\vec{r}_0)$ $\frac{1}{(2\pi)^3\lvert\vec{k}\rvert^3}\left(\sin(\lvert\vec{k}\rvert R)-\lvert\vec{k}\rvert R\cos(\lvert\vec{k}\rvert R)\right)\exp(-i\vec{k}\cdot\vec{r}_0)$	$\frac{2\pi R}{\lvert\vec{k}\rvert}J_1(\lvert\vec{k}\rvert R)\exp(-i\vec{k}\cdot\vec{r}_0)$ $\frac{4\pi}{\lvert\vec{k}\rvert^3}\left(\sin(\lvert\vec{k}\rvert R)-\lvert\vec{k}\rvert R\cos(\lvert\vec{k}\rvert R)\right)\exp(-i\vec{k}\cdot\vec{r}$

Here $H(x)$ is the Heaviside step function, $\delta(x)$ is the Dirac delta function, $J_1(x)$ is the first order Bessel function of the first kind, and d is the number of dimensio

Calculate a Fourier transform numerically.

http://lamp.tu-graz.ac.at/~hadley/ss1/crystaldiffraction/ft/ft.php

Properties of Fourier transforms

Linearity and superposition

$\mathcal{F}\{\alpha f(\vec{r}) + \beta g(\vec{r})\} = \alpha\mathcal{F}\{f(\vec{r})\} + \beta\mathcal{F}\{g(\vec{r})\}$ where α and β are any constants.

Similarity

$\mathcal{F}\{f\left(\frac{\vec{r}}{a}\right)\} = |a|^d\mathcal{F}\{f(\vec{r})\}.$

Shift

$\mathcal{F}\{f(\vec{r} - \vec{r}_0)\} = \mathcal{F}\{f(\vec{r})\}\exp\left(-i\vec{k}\cdot\vec{r}_0\right).$

Convolution (Faltung)

$$f(\vec{r}) * g(\vec{r}) = \int f(\vec{r}')g(\vec{r}-\vec{r}')d\vec{r}$$

Notation [-1,-1]: $\mathcal{F}\{fg\} = \mathcal{F}\{f\} * \mathcal{F}\{g\}, \quad \mathcal{F}^{-1}\{FG\} = \frac{1}{2\pi}\mathcal{F}^{-1}\{F\} * \mathcal{F}^{-1}\{G\}.$

Notation [1,-1]: $\mathcal{F}\{fg\} = \frac{1}{2\pi}\mathcal{F}\{f\} * \mathcal{F}\{g\}, \quad \mathcal{F}^{-1}\{FG\} = \mathcal{F}^{-1}\{F\} * \mathcal{F}^{-1}\{G\}.$

Notation [0,-1]: $\mathcal{F}\{fg\} = \frac{1}{\sqrt{2\pi}}\mathcal{F}\{f\} * \mathcal{F}\{g\}, \quad \mathcal{F}^{-1}\{FG\} = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\{F\} * \mathcal{F}^{-1}\{G\}.$

Notation [0,-2π]: $\mathcal{F}\{fg\} = \mathcal{F}\{f\} * \mathcal{F}\{g\}, \quad \mathcal{F}^{-1}\{FG\} = \mathcal{F}^{-1}\{F\} * \mathcal{F}^{-1}\{G\}.$

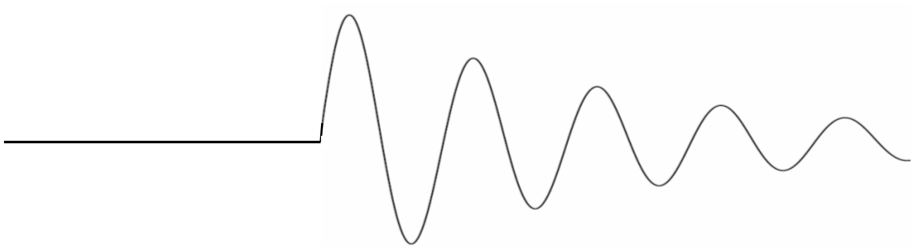
Impulse response function (Green's functions)

A Green's function is the solution to a linear differential equation for a δ-function driving force

For instance, $$m\frac{d^2g}{dt^2} + b\frac{dg}{dt} + kg = \delta(t)$$

has the solution

$$g(t) = \frac{1}{m}\exp\left(\frac{-bt}{2m}\right)\sin\left(\frac{\sqrt{4mk-b^2}}{2m}t\right) \qquad t > 0$$



Green's functions

A driving force *f* can be thought of a being built up of many delta functions after each other.

$$f(t) = \int \delta(t-t') f(t') dt'$$

By superposition, the response to this driving function is superposition,

$$u(t) = \int g(t-t') f(t') dt'$$

For instance, $$m\frac{d^2u}{dt^2} + b\frac{du}{dt} + ku = f(t)$$

has the solution

$$u(t) = \int_{-\infty}^{\infty} \frac{1}{m} \exp\left(\frac{-b(t-t')}{2m}\right) \sin\left(\frac{\sqrt{4mk-b^2}}{2m}(t-t')\right) f(t') dt'$$

Green's function converts a differential equation into an integral equation

Generalized susceptibility

A driving function *f* causes a response *u*

If the driving force is sinusoidal,

$$f(t) = F_0 e^{i\omega t}$$

The response will also be sinusoidal.

$$u(t) = \int g(t-t') f(t') dt' = \int g(t-t') F_0 e^{i\omega t'} dt'$$

The generalized susceptibility at frequency ω is

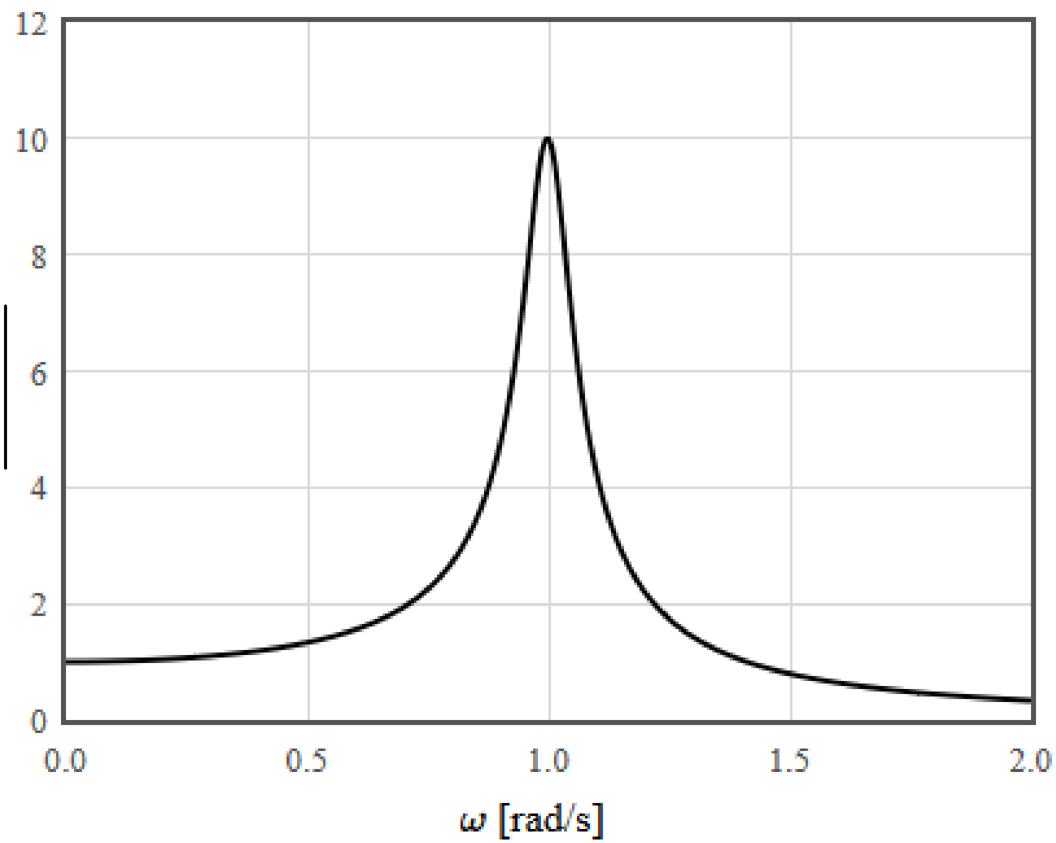
$$\chi(\omega) = \frac{u}{f} = \frac{\int g(t-t') e^{i\omega t'} dt'}{e^{i\omega t}}$$

Generalized susceptibility

$m =$ 1 [kg] $b =$ 0.1 [N s/m] $k =$ 1 [N/m] Plot

$Q = \frac{\sqrt{mk}}{b} =$ 10

$$|\chi(\omega)| = \left|\frac{u}{f}\right|$$



http://lampx.tugraz.at/~hadley/physikm/apps/resonance.en.php

Generalized susceptibility

$$\chi(\omega) = \frac{u}{f} = \frac{\int g(t-t')e^{i\omega t'}dt'}{e^{i\omega t}}$$

Since the integral is over t', the factor with t can be put in the integral.

$$\chi(\omega) = \int g(t-t')e^{-i\omega(t-t')}dt'$$

Change variables to $\tau = t - t'$, $d\tau = -dt'$, reverse the limits of integration

$$\chi(\omega) = \int g(\tau)e^{-i\omega\tau}d\tau$$

The susceptibility is the Fourier transform of the Green's function.

$$g(t) = \frac{1}{2\pi}\int \chi(\omega)e^{i\omega t}d\omega \qquad F_{1,-1}$$

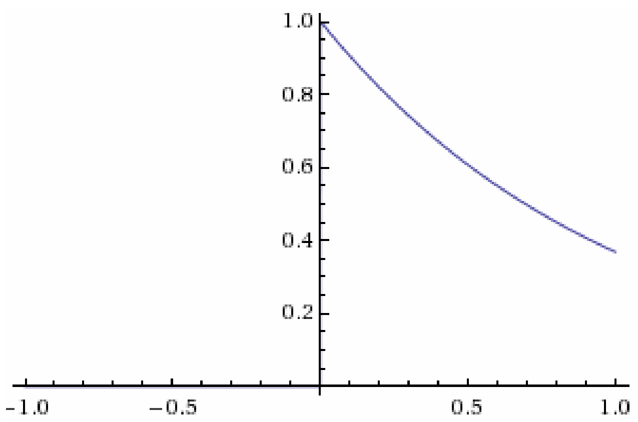
First order differential equation

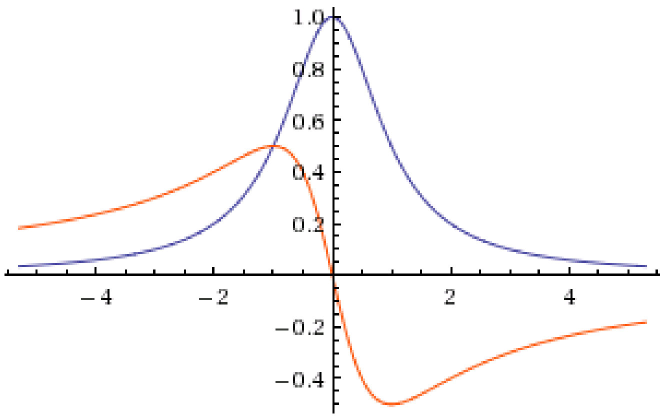
$$m\frac{dg}{dt} + bg = \delta(t)$$

$$g(t) = \frac{1}{m}H(t)\exp\left(-\frac{bt}{m}\right) \qquad \frac{b}{m} > 0$$

$$\chi(\omega) = \int g(t)e^{-i\omega t}dt$$

$$\chi(\omega) = \frac{1}{m}\frac{\frac{b}{m} - i\omega}{\left(\frac{b}{m}\right)^2 + \omega^2}$$





The Fourier transform of a decaying exponential is a Lorentzian

Susceptibility

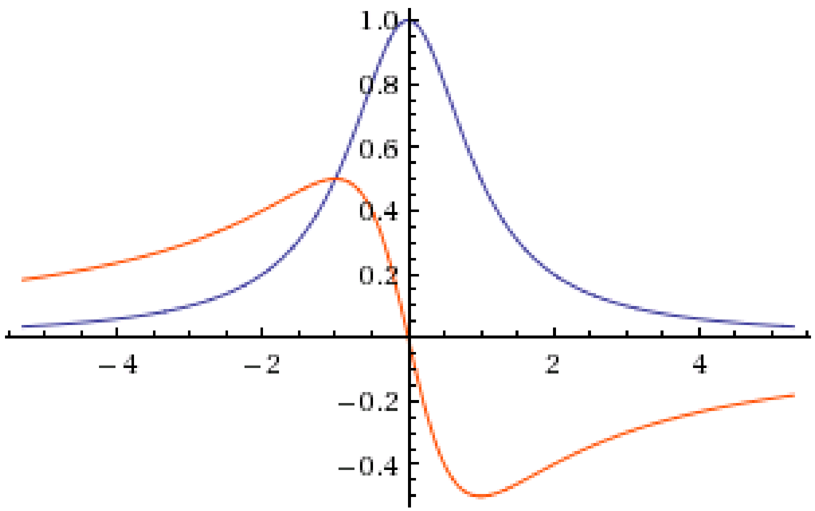
$$m\frac{du}{dt}+bu=F(t)$$

Assume that *u* and *F* are sinusoidal $u = Ae^{i\omega t}$ $F = F_0 e^{i\omega t}$

$$i\omega mA + bA = F_0$$

$$A=\frac{F_0}{b+i\omega m}=F_0\frac{b-i\omega m}{b^2+m^2\omega^2}$$

$$\chi=\frac{u}{F}=\frac{1}{m}\frac{\frac{b}{m}-i\omega}{\left(\frac{b}{m}\right)^2+\omega^2}$$



The sign of the imaginary part depends on whether you use $e^{i\omega t}$ or $e^{-i\omega t}$.

Susceptibility

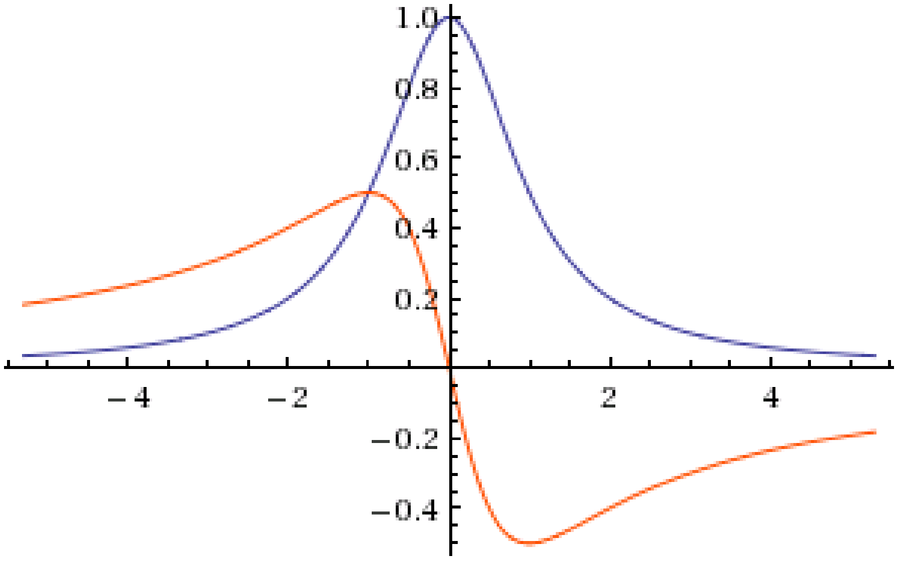
$$m\frac{dg}{dt}+bg=\delta(t)$$

Fourier transform the differential equation

$$i\omega m\chi(\omega)+b\chi(\omega)=1$$

$$\chi=\frac{1}{b+i\omega m}$$

$$\chi=\frac{1}{m}\frac{\frac{b}{m}-i\omega}{\left(\frac{b}{m}\right)^2+\omega^2}$$

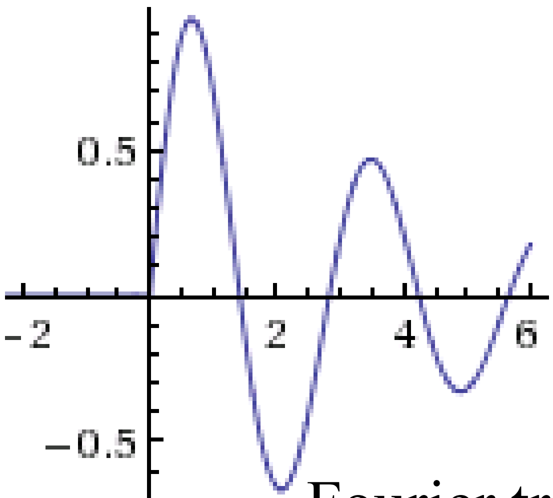


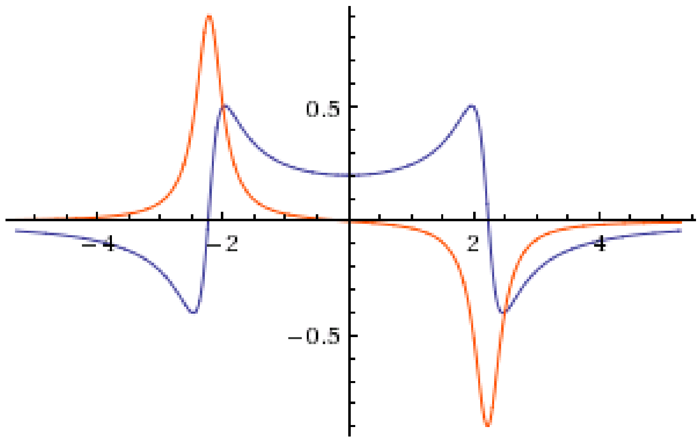
Damped mass-spring system

$$m\frac{d^2g}{dt^2} + b\frac{dg}{dt} + kg = \delta(t)$$

$$-\omega^2 m\chi + i\omega b\chi + k\chi = 1$$

$$g = e^{\lambda t} \qquad \lambda_{\pm} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$





Fourier transform pair

$$\chi = \left(\frac{1}{m}\right)\frac{\frac{k}{m} - \omega^2 - i\omega\frac{b}{m}}{\left(\frac{k}{m} - \omega^2\right)^2 + \left(\omega\frac{b}{m}\right)^2}$$

$$g(t) = H(t)\frac{1}{m}\exp\left(\frac{-bt}{2m}\right)\sin\left(\frac{\sqrt{4mk - b^2}}{2m}t\right)$$

More complex linear systems

Any coupled system of linear differential equations can be written as a set of first order equations

$$\frac{d\vec{x}}{dt} = M\vec{x}$$

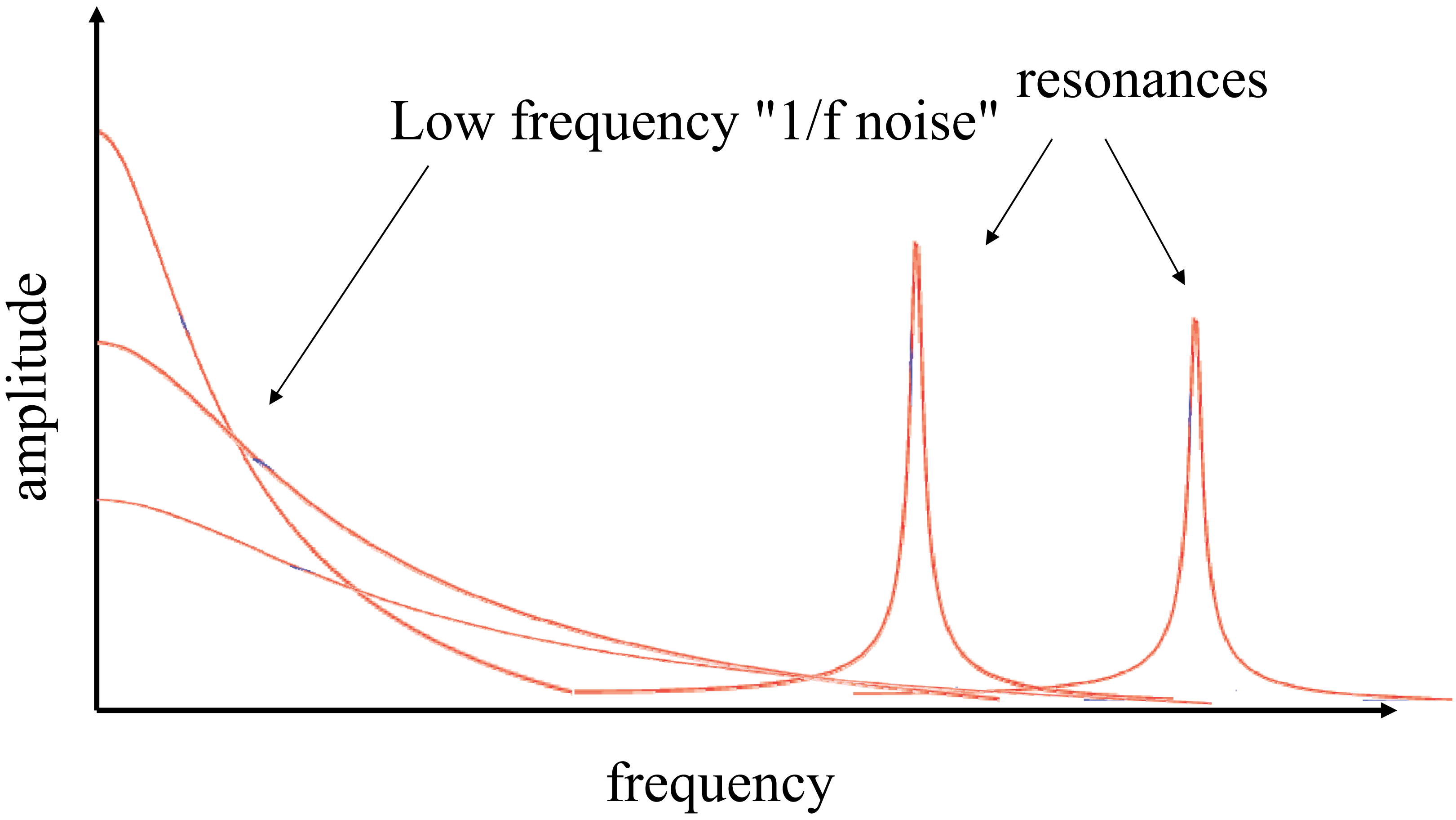
The solutions have the form $\vec{x}_i e^{\lambda_i t}$

where $\vec{x}_i$ are the eigenvectors and λ_i are the eigenvalues of matrix *M*.

$\mathrm{Re}(\lambda_i) < 0$ for stable systems

λ_i is either real and negative (overdamped) or comes in complex conjugate pairs with a negative real part (underdamped).

More complex linear systems



Odd and even components

Any function $f(t)$ can be written in terms of its odd and even components

$$E(t) = \tfrac{1}{2}[f(t) + f(-t)]$$

$$O(t) = \tfrac{1}{2}[f(t) - f(-t)]$$

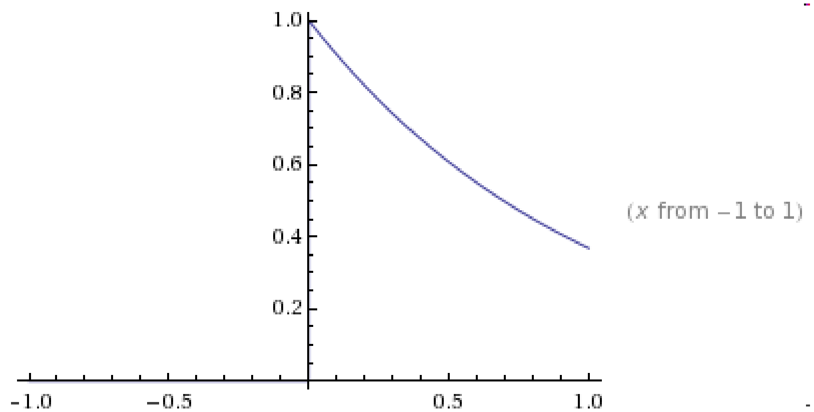
$$f(t) = E(t) + O(t)$$

$$f(t) = \tfrac{1}{2}[f(t) + f(-t)] + \tfrac{1}{2}[f(t) - f(-t)]$$

$$\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \left(E(t) + O(t)\right)\left(\cos\omega t - i\sin\omega t\right)dt$$

$$= \int_{-\infty}^{\infty} E(t)\cos\omega t dt - i\int_{-\infty}^{\infty} O(t)\sin\omega t dt$$

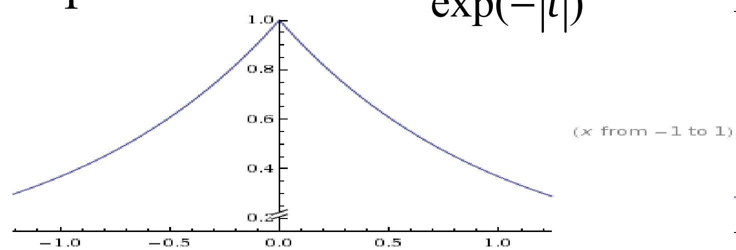
The Fourier transform of $E(t)$ is real and even
The Fourier transform of $O(t)$ is imaginary and odd

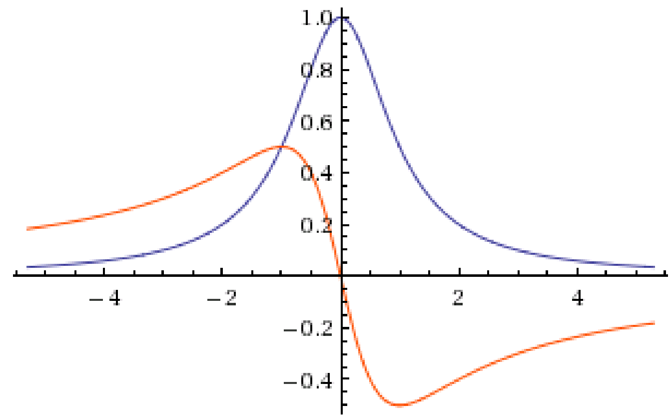


$$\chi(\omega) = \frac{1}{m}\frac{\frac{b}{m} - i\omega}{\left(\frac{b}{m}\right)^2 + \omega^2}$$

even component

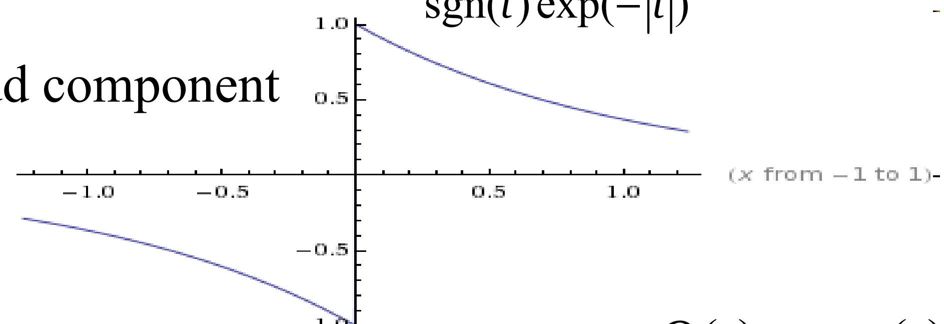
$\exp(-|t|)$





odd component

$\text{sgn}(t)\exp(-|t|)$



$$O(t) = \text{sgn}(t)E(t)$$

$$E(t) = \text{sgn}(t)O(t)$$

Causality and the Kramers-Kronig relations (I)

$$\chi(\omega) = \int g(\tau) e^{-i\omega\tau} d\tau = \int E(\tau)\cos(\omega\tau) d\tau - i\int O(\tau)\sin(\omega\tau) d\tau = \chi'(\omega) + i\chi''(\omega)$$

The real and imaginary parts of the susceptibility are related.

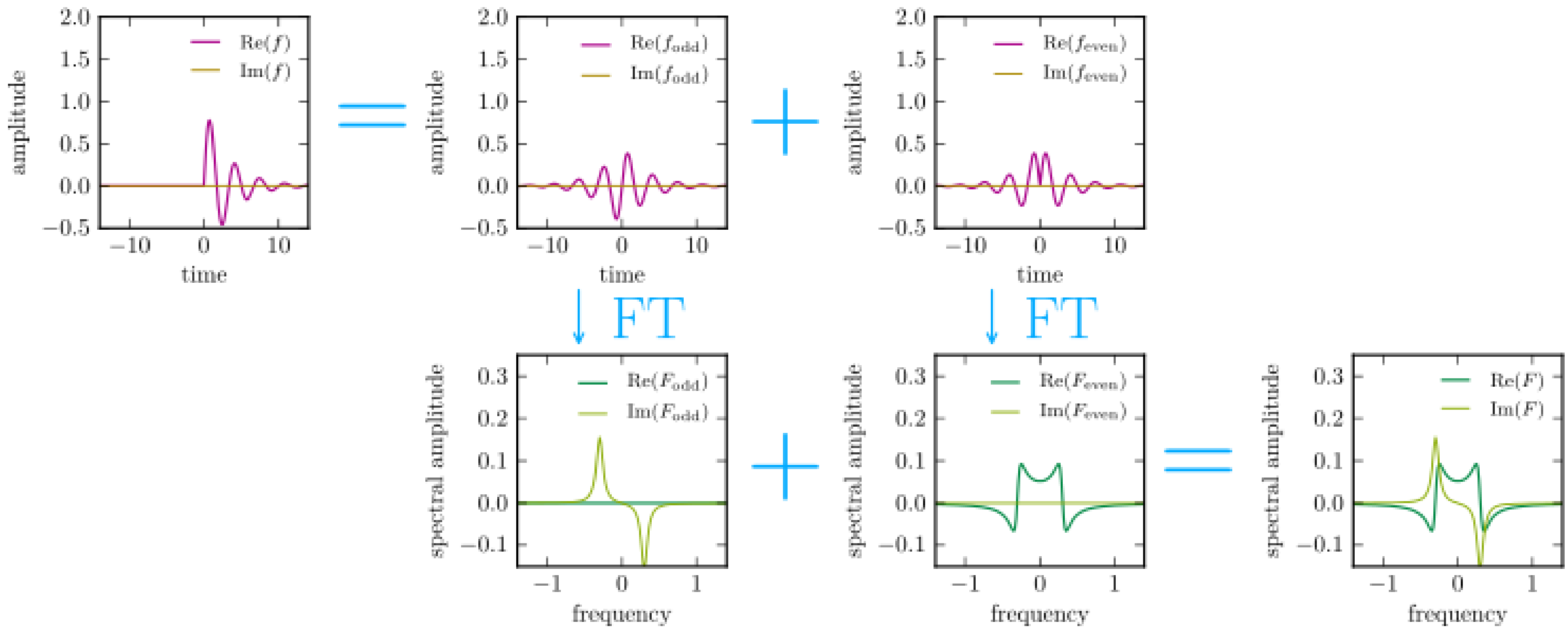
If you know χ', inverse Fourier transform to find $E(t)$. Knowing $E(t)$ you can determine $O(t) = \mathrm{sgn}(t)E(t)$. Fourier transform $O(t)$ to find χ".

$$\chi'(\omega) = \int_{-\infty}^{\infty} E(t)\cos(\omega t) dt \qquad E(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi'(\omega)\cos(\omega t) d\omega$$

$$\boxed{O(t) = \mathrm{sgn}(t)E(t) \qquad E(t) = \mathrm{sgn}(t)O(t)}$$

$$\chi''(\omega) = -\int_{-\infty}^{\infty} O(t)\sin(\omega t) dt \qquad O(t) = \frac{-1}{2\pi}\int_{-\infty}^{\infty} \chi''(\omega)\sin(\omega t) d\omega$$

Kramers-Kronig relations



If you know any of these for just positive frequencies,
you can calculate all the others.

https://en.wikipedia.org/wiki/Kramers%E2%80%93Kronig_relations

Causality and the Kramers-Kronig relation (II)

Real space	Reciprocal space
$E(t) = \mathrm{sgn}(t)O(t)$ $O(t) = \mathrm{sgn}(t)E(t)$	$\chi'(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\chi''(\omega')}{\omega' - \omega} d\omega'$ $\chi''(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\chi'(\omega')}{\omega' - \omega} d\omega'$

$$\chi' = \frac{-i}{\pi\omega} * i\chi'', \quad i\chi'' = \frac{-i}{\pi\omega} * \chi'$$

Take the Fourier transform, use the convolution theorem.

P: Cauchy principle value (go around the singularity and take the limit as you pass by arbitrarily close)

Singularity makes a numerical evaluation more difficult.

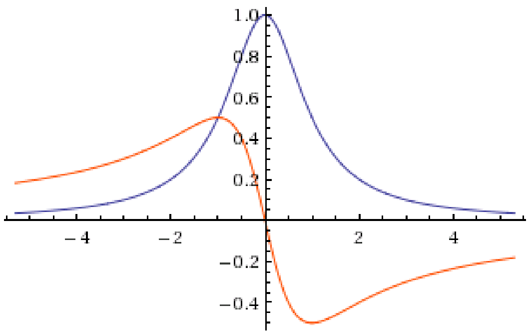
http://lamp.tu-graz.ac.at/~hadley/ss2/linearresponse/causality.php

Kramers-Kronig relations (III)

$$\chi''(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\chi'(\omega')}{\omega' - \omega} d\omega'$$

$$\chi'(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\chi''(\omega')}{\omega' - \omega} d\omega'$$

Kramers-Kronig relations II



$$\chi'(\omega) = \chi'(-\omega)$$
$$\chi''(\omega) = -\chi''(-\omega)$$

Real part is even
Imaginary part is odd

$$\chi'(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{0} \frac{\chi''(\omega')}{\omega' - \omega} d\omega' - \frac{1}{\pi} P \int_{0}^{\infty} \frac{\chi''(\omega')}{\omega' - \omega} d\omega'$$

change variables ω'→-ω'
(4 minus signs)

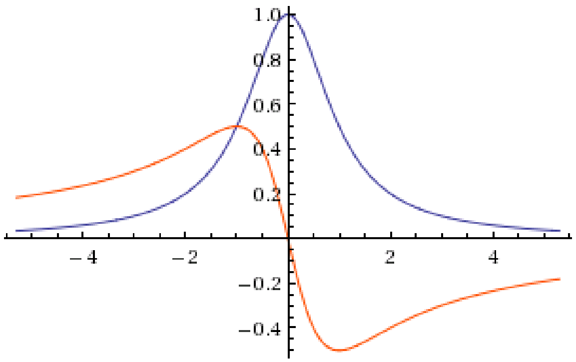
Kramers-Kronig relations (III)

$$\chi'(\omega) = -\frac{1}{\pi} P\int_0^\infty \frac{\chi''(\omega')}{\omega'+\omega} d\omega' - \frac{1}{\pi} P\int_0^\infty \frac{\chi''(\omega')}{\omega'-\omega} d\omega'$$

$$\frac{1}{\omega'+\omega} + \frac{1}{\omega'-\omega} = \frac{2\omega'}{(\omega')^2 - \omega^2}$$

$$\chi'(\omega) = \frac{2}{\pi} P\int_0^\infty \frac{\omega'\chi''(\omega')}{(\omega')^2 - \omega^2} d\omega'$$

$$\chi''(\omega) = -\frac{2}{\pi} P\int_0^\infty \frac{\omega\chi'(\omega')}{(\omega')^2 - \omega^2} d\omega'$$



Singularity is stronger in this form.