18. Optical Properties of Insulators

Dec. 5, 2019

Causality and the Kramers-Kronig relations (I)

$$\chi(\omega) = \int g(\tau)e^{-i\omega\tau}d\tau = \int E(\tau)\cos(\omega\tau)d\tau - i\int O(\tau)\sin(\omega\tau)d\tau = \chi'(\omega) + i\chi''(\omega)$$

The real and imaginary parts of the susceptibility are related.

If you know χ', inverse Fourier transform to find $E(t)$. Knowing $E(t)$ you can determine $O(t)$ = sgn(t)$E(t)$. Fourier transform $O(t)$ to find χ".

$$\chi'(\omega) = \int_{-\infty}^{\infty} E(t)\cos(\omega t)dt \qquad E(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi'(\omega)\cos(\omega t)d\omega$$

$$\boxed{O(t) = \mathrm{sgn}(t)E(t) \qquad E(t) = \mathrm{sgn}(t)O(t)}$$

$$\chi''(\omega) = -\int_{-\infty}^{\infty} O(t)\sin(\omega t)dt \qquad O(t) = \frac{-1}{2\pi}\int_{-\infty}^{\infty} \chi''(\omega)\sin(\omega t)d\omega$$

Impulse response/generalized susceptibility

The impulse response function is the response of the system to a δ-function excitation. The response function must be zero before the excitation.

The generalized susceptibility is the Fourier transform of the impulse response function.

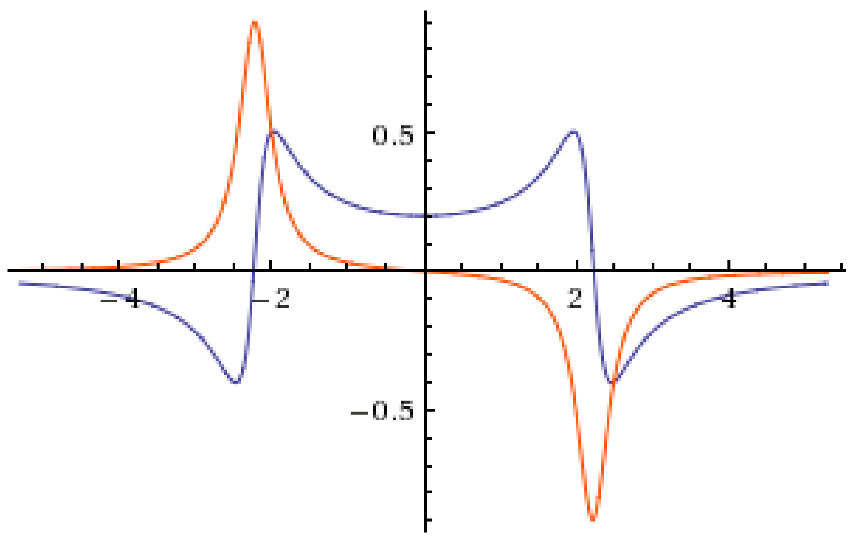
Any function that is zero before the excitation and nonzero afterwards must have both an odd component and an even component.

The generalized susceptibility must have a real and imaginary part. All information about the real part is contained in the imaginary part and vice versa.

Fluctuation-dissipation theorem

The fluctuation-dissipation theorem relates the size of the fluctuations to the dissipation in a system.

Most of the dissipation in a resonant system occurs at frequencies near the resonance.



http://en.wikipedia.org/wiki/Fluctuation_dissipation_theorem

Fluctuation-dissipation theorem

Brownian motion: The response to thermal noise is related to the viscosity.

$$m\frac{dv}{dt} = -\mu v \qquad\qquad D = \mu k_B T$$

Johnson noise: The voltage fluctuations are related to the resistance.

$$V_{rms} = \sqrt{4k_B TRB}$$

The fluctuation-dissipation theorem holds at equilibrium (where the equations are linear to a good approximation).

http://en.wikipedia.org/wiki/Fluctuation_dissipation_theorem

Dielectric response of insulators

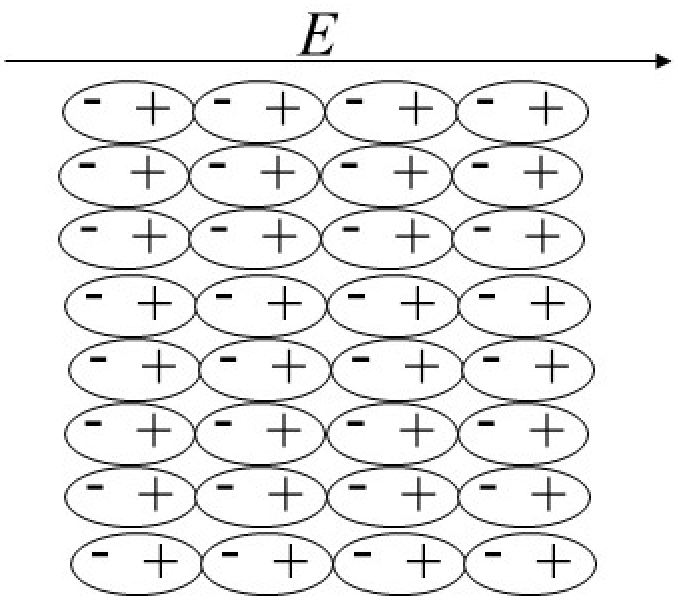
The electric polarization is related to the electric field

$$P_i = \varepsilon_0 \chi_{ij} E_j$$

The electric displacement vector *D* is also related to the electric field

$$D_i = P_i + \varepsilon_0 E_i = \varepsilon_0 (1 + \chi_{ij}) E_j = \varepsilon_0 \varepsilon_{ij} E_j$$

$$\varepsilon_{ij} = (1 + \chi_{ij})$$



E is decreased by a factor of the dielectric constant

Dielectric response of insulators

In an insulator, charge is bound. The response to an electric field can be modeled as a collection of damped harmonic oscillators

$$P = nex$$

Macroscopic polarization → P

density → n

ex = dipole moment

The core electrons of a metal respond to an electric field like this too.

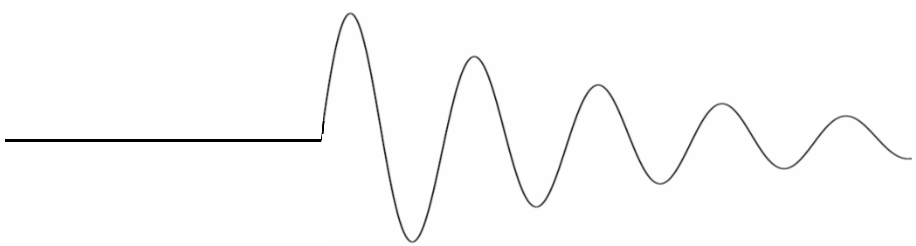
Dielectric response of insulators

The differential equation that describes how the position of the charge changes in time is:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = -eE(t)$$

The impulse response function is:

$$g(t) = -\frac{1}{b}\exp\left(\frac{-bt}{2m}\right)\sin\left(\frac{\sqrt{4mk-b^2}}{2m}t\right) \qquad t > 0$$



Electric susceptibility

$$\vec{P} = \varepsilon_0 \chi_E \vec{E} \qquad \vec{P} = nq\vec{x}$$

$$\chi_E = \frac{P}{\varepsilon_0 E} = \frac{nqx}{\varepsilon_0 E}$$

Assume a solution of the form $x(\omega)e^{i\omega t}$, $E(\omega)e^{i\omega t}$

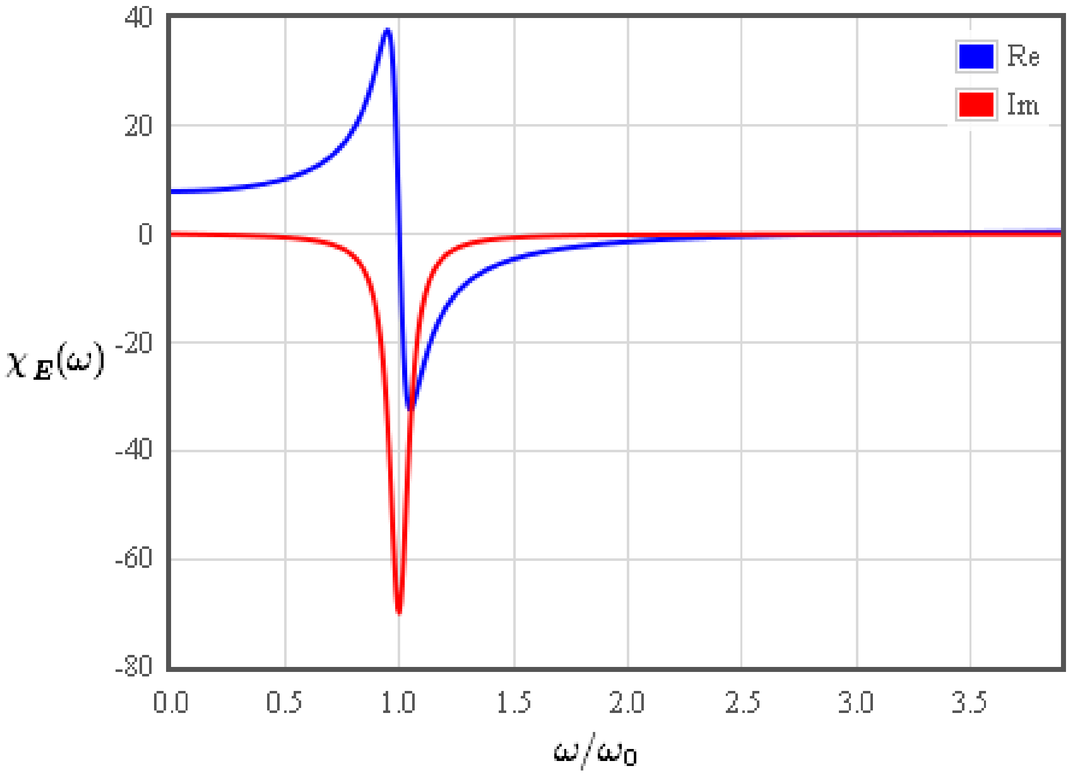
$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = qE(t)$$

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = -\frac{qE}{m}$$

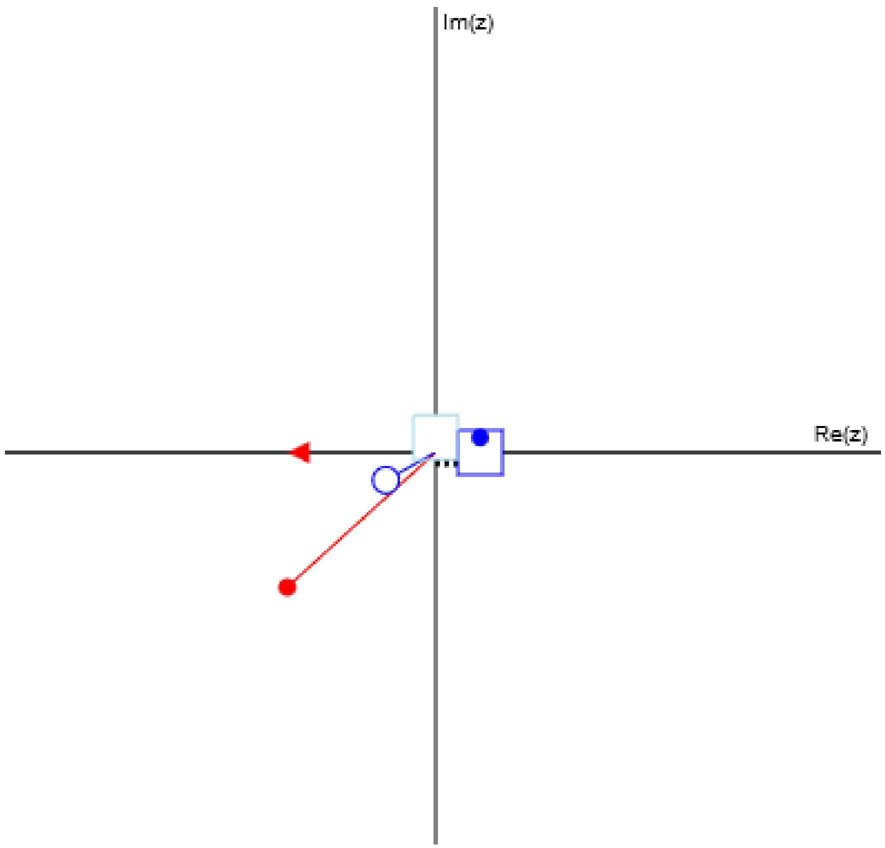
$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \gamma = \frac{b}{m}$$

Electric susceptibility

$$\chi_E(\omega) = \frac{n_{\omega_0} q^2}{\varepsilon_0 m} \frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega}$$



Resonance of a damped driven harmonic oscillator



m =4 [kg]

b =1 [N s/m]

k =6 [N/m]

F_0 =0.9 [N]

ω =0.8 [rad/s]

$$\omega_0 = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = 1.22 \text{ [rad/s]} = 0.194 \text{ [Hz]}$$

$$\theta = \text{atan}\left(\frac{\omega b}{k - m\omega^2}\right) = 0.228 \text{ [rad]} = 13.1 \text{ [deg]}$$

$$|A| = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \omega^2 b^2}} = 0.255 \text{ [m]}$$

$$Q = \frac{\sqrt{mk}}{b} = 4.90$$

Display $F_0 e^{i\omega t}$: ☑ Display $|A| e^{i(\omega t - \theta)}$: ☑

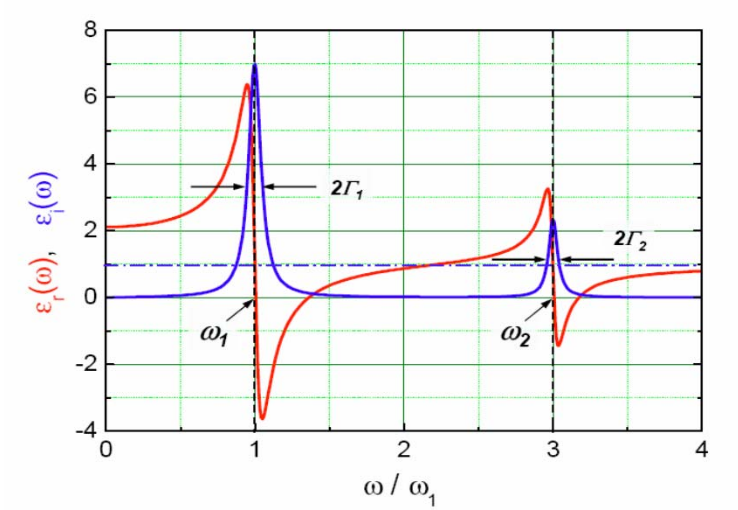
Display transients z: ☑ Display x_2: ☑

http://lamp.tu-graz.ac.at/~hadley/physikm/apps/resonance.en.php

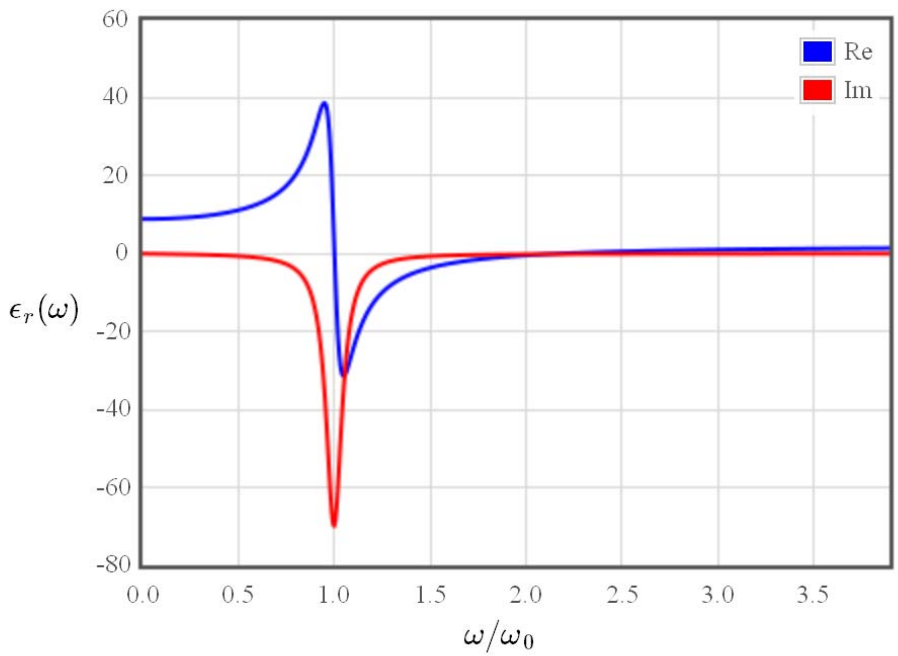
Dielectric function

$$\vec{D} = \epsilon_r \epsilon_0 \vec{E} = \vec{P} + \epsilon_0 \vec{E}$$

$$\varepsilon_r(\omega) = 1 + \chi_E(\omega) = 1 + \frac{n_{\omega_0} q^2}{\varepsilon_0 m} \frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

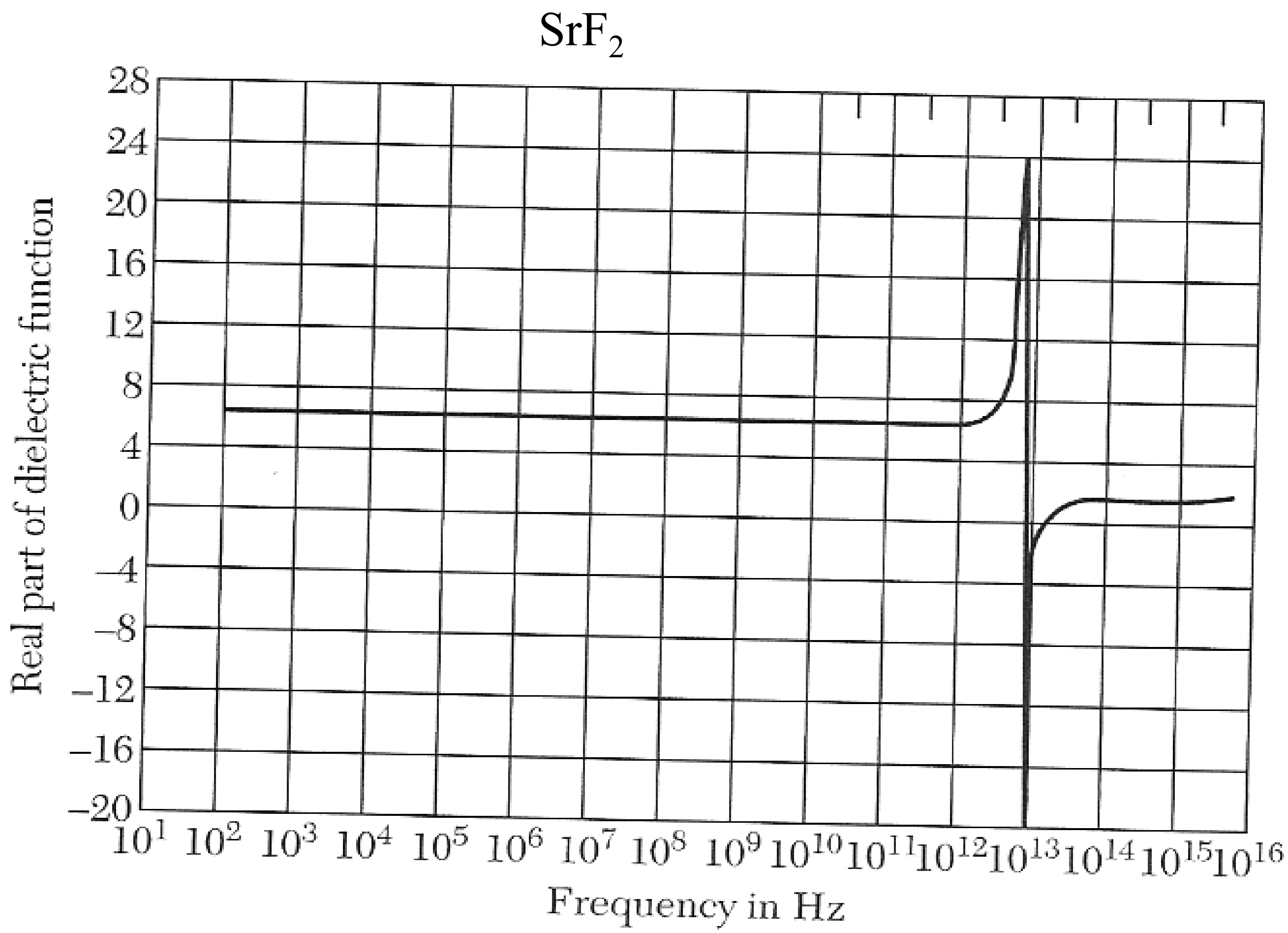


Gross and Marx



There can be more resonances.

Dielectric function of insulators



Insulators can often be modeled as a simple resonance.

Dispersion relation

In the section on photons, we derived the wave equation for light in vacuum. Here the wave equation for light in a dielectric material is derived.

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

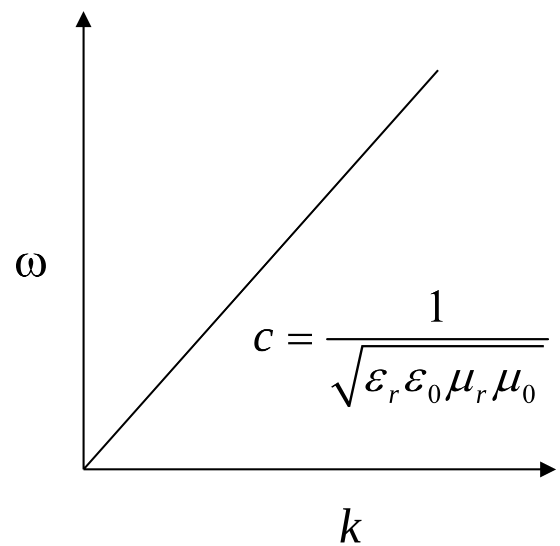
$$\vec{D} = \varepsilon \vec{E} \qquad \vec{B} = \mu \vec{H}$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

Take the curl $$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial \nabla \times \vec{H}}{\partial t}$$

$$\nabla \left(\cancelto{0}{\nabla \cdot \vec{E}} \right) - \nabla^2 \vec{E} = -\mu \frac{\partial^2 \vec{D}}{\partial t^2}$$

$$\varepsilon_r \varepsilon_0 \mu_r \mu_0 \frac{\partial^2 \vec{D}}{\partial t^2} = \nabla^2 \vec{D}$$



The normal mode solutions are plane waves: $\vec{D} = \vec{D}_0 \exp\left(i \left(\vec{k} \cdot \vec{r} - \omega t \right) \right)$

$$\boxed{\varepsilon(\omega, k) \mu_0 \varepsilon_0 \omega^2 = k^2}$$

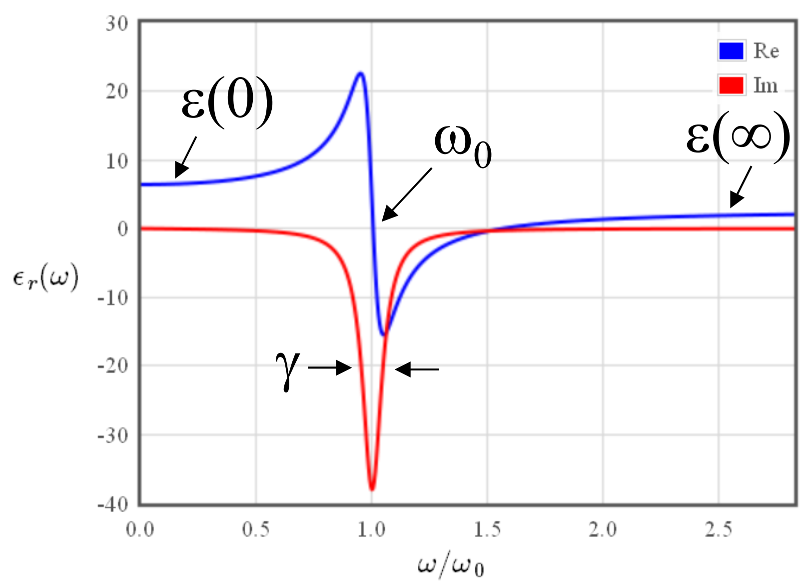
Dispersion relation

$$\varepsilon(\omega)\mu_0\varepsilon_0\omega^2 = k^2$$

If ε is real and positive: propagating electromagnetic waves $\exp\left(i\left(\vec{k}\cdot\vec{r}-\omega t\right)\right)$

If $\varepsilon_r < 0$: decaying solutions $\exp(-\vec{k}\cdot\vec{r}-i\omega t)$

If ε is complex, $\varepsilon_r > 0$: decaying electromagnetic waves $\exp\left(i\left(\vec{k}\cdot\vec{r}-\omega t\right)\right)\exp(-\kappa r)$



Dielectric function

Dispersion relation: $\varepsilon_r \mu_0 \varepsilon_0 \omega^2 = k^2$

$$k = \sqrt{\varepsilon_r \mu_0 \varepsilon_0}\,\omega = \frac{\sqrt{\varepsilon_r}\,\omega}{c}$$

Measurable: $\sqrt{\varepsilon} = n + iK$

refractive index (n) — extinction coefficient (K)

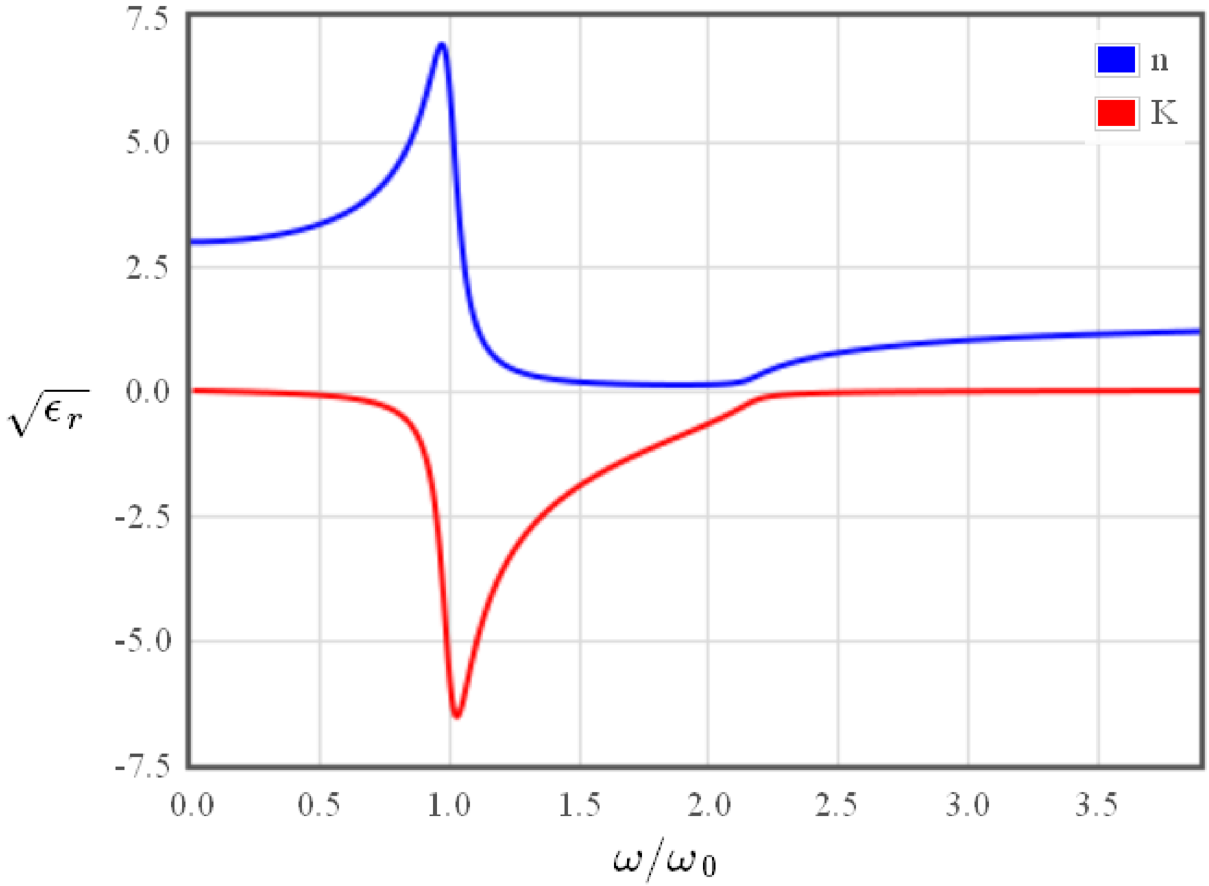
Vacuum	Dielectric
$\omega = ck$	$k = \dfrac{\sqrt{\varepsilon_r}\,\omega}{c}$
$\exp\left(i\left(\dfrac{\omega x}{c} - \omega t\right)\right)$	$\exp\left(i\left(\dfrac{(n+iK)\omega x}{c} - \omega t\right)\right)$ $= \exp\left(i\left(\dfrac{n\omega x}{c} - \omega t\right)\right)\exp\left(-\dfrac{\omega K x}{c}\right)$

Intensity $I(x) = I(0)\exp(-\alpha x)$ J m^{-2} s^{-1} Beer-Lambert

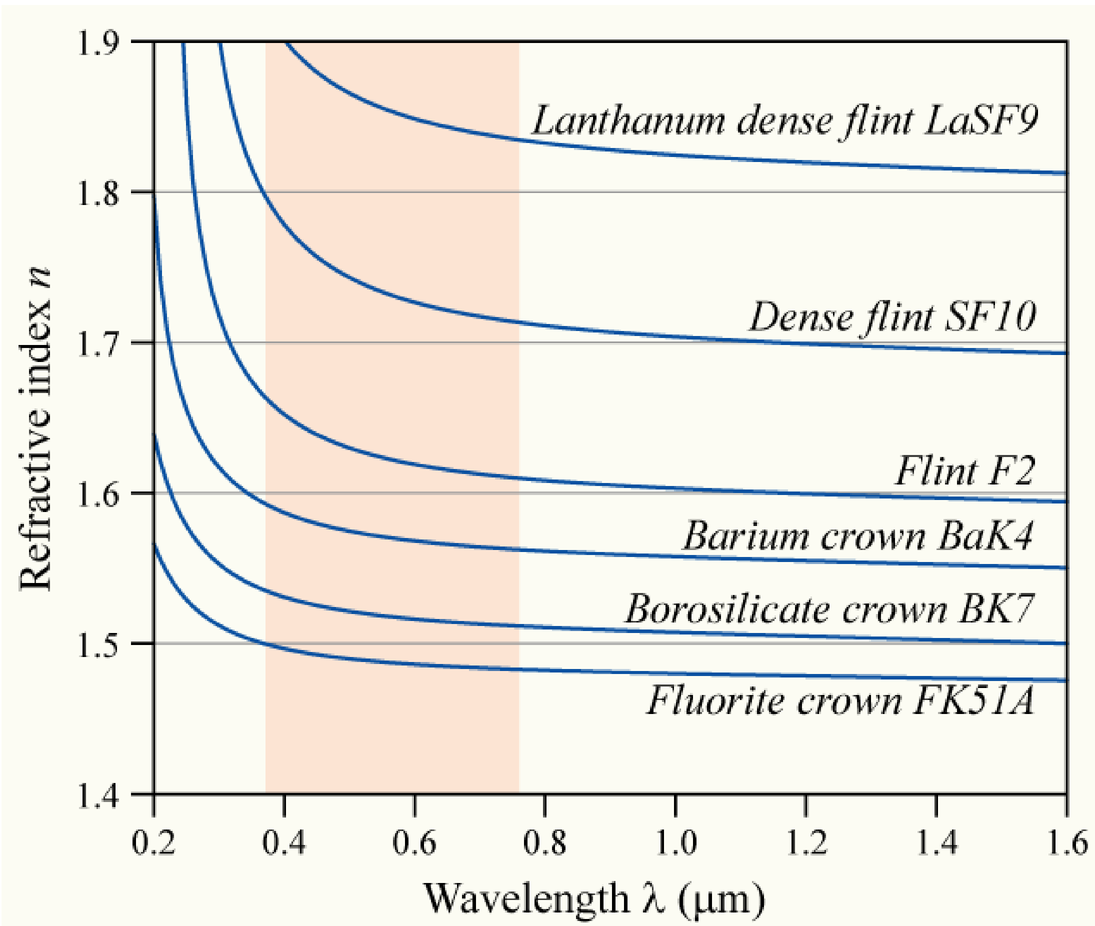
absorption coefficient ⟶ $\alpha = \dfrac{2\omega K}{c}$

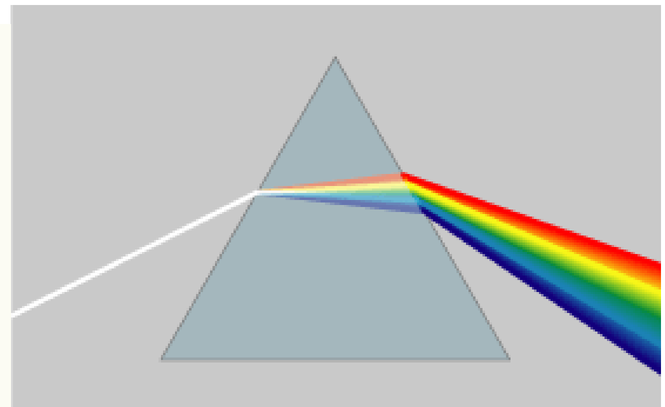
The index of refraction *n* and the extinction coefficient *K*

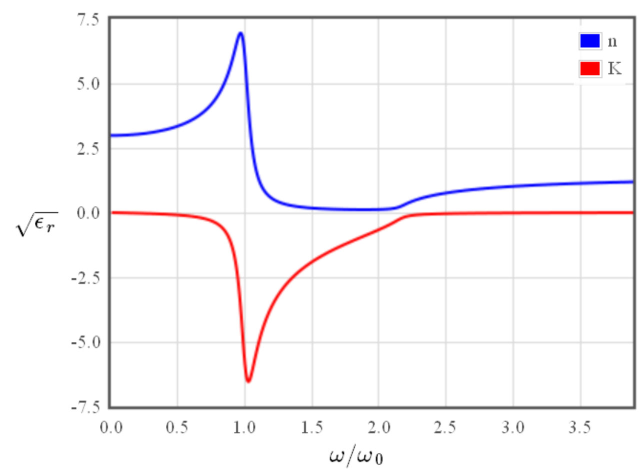
$$\sqrt{\epsilon_r} = n + iK$$



Dispersion







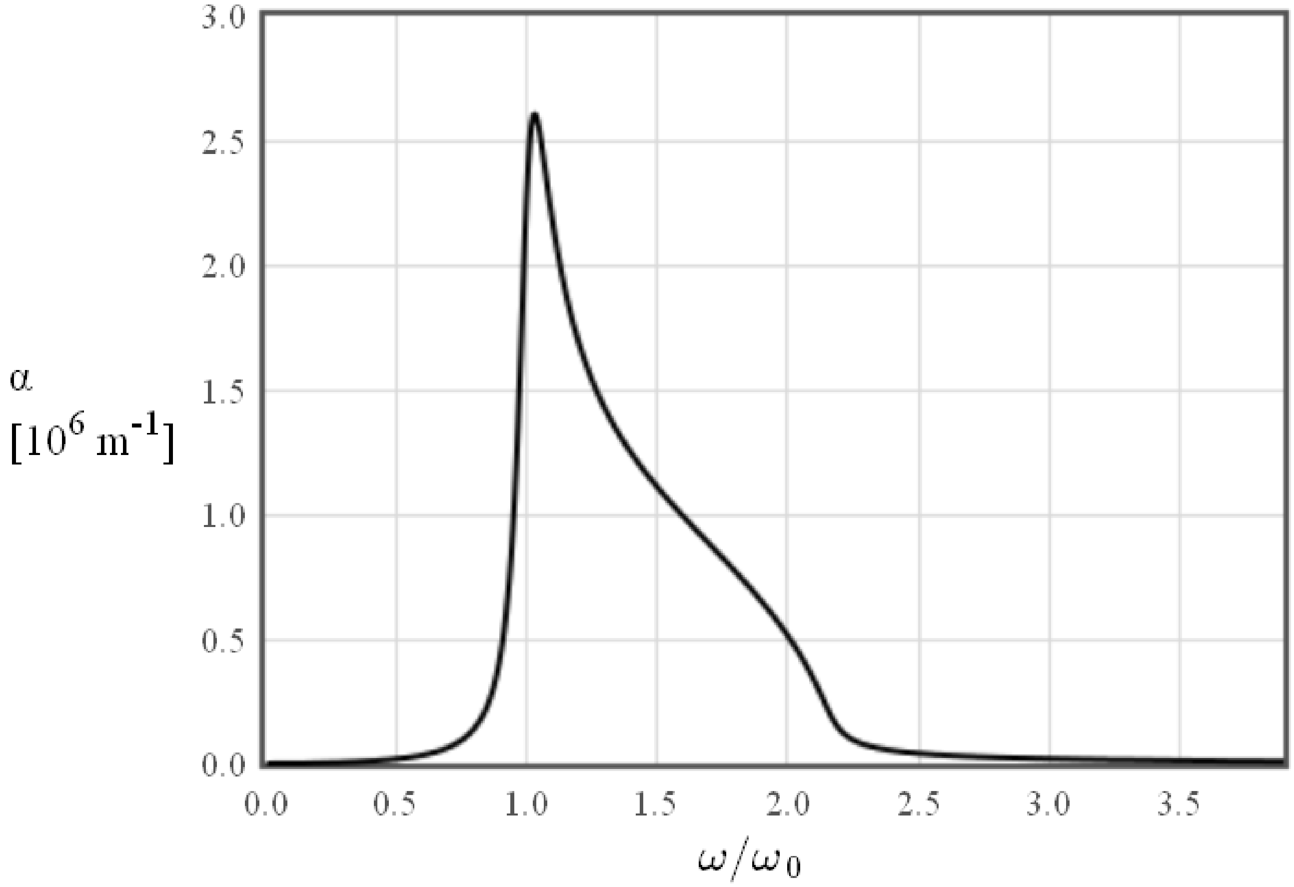
Cause of chromatic aberration in lenses.

http://en.wikipedia.org/wiki/Dispersion_%28optics%29#mediaviewer/File:Prism_rainbow_schema.png
http://en.wikipedia.org/wiki/Refractive_index

Absorption coefficient α

$$I = I_0 \exp(-\alpha x)$$

$$\alpha = \frac{2\omega K}{c}$$



Reflectance

$$R = \frac{(n-1)^2 + K^2}{(n+1)^2 + K^2}$$