

# 11. Free Electrons in a Magnetic Field

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Nov 7, 2019

# Charged particle in a magnetic field

$$\vec{F} = -e\vec{v} \times \vec{B} = ma$$

$$evB_z = \frac{mv^2}{R}$$

solve for velocity

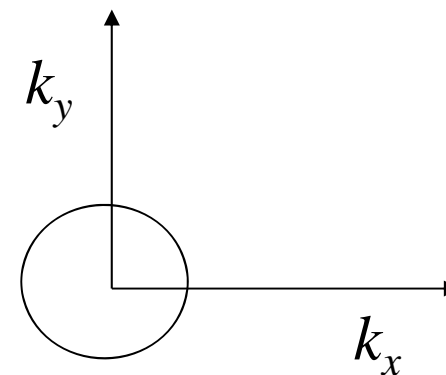
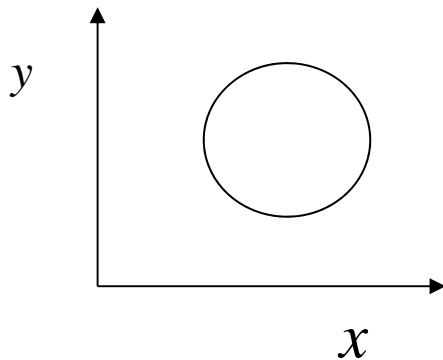
$$v = \frac{eB_z R}{m}$$

$$v = \omega_c R$$

$$\omega_c = \frac{eB_z}{m}$$



Magnetron

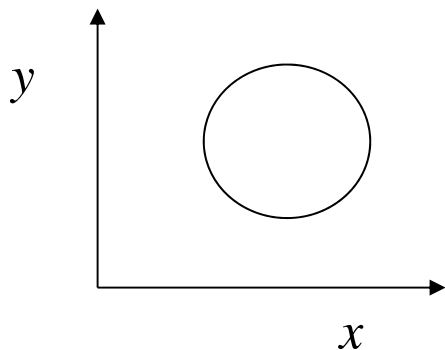


# Why don't metals in a B field radiate?

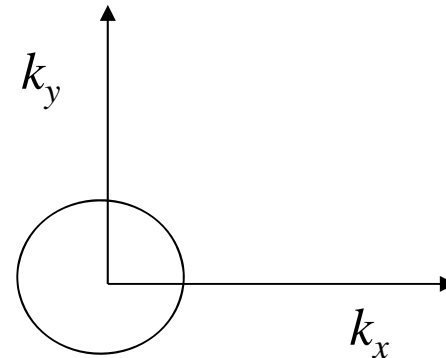
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Magnetron



$$\omega_c = \frac{eB_z}{m}$$



There are no lower lying states to fall into. We need a quantum description.

$$\frac{-\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

Magnetic force depends on the velocity, not on the position.

# Electrons in a magnetic field

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Lorentz force law  $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

Euler Lagrange equations  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} = 0$

Lagrangian:  $L = \frac{1}{2} m v^2 - qV(\vec{r}, t) + q\vec{v} \cdot \vec{A}(\vec{r}, t)$

Kittel: Appendix G

<http://lamp.tu-graz.ac.at/~hadley/ss2/IQHE/cpimf.php>

# Lagrangian

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$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} = 0$$

$$L = \frac{1}{2} m v^2 - qV(\vec{r}, t) + q\vec{v} \cdot \vec{A}(\vec{r}, t)$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v_x} \right) = \frac{d}{dt} (m v_x + q A_x) = m \frac{d v_x}{dt} + q \frac{d A_x}{dt}$$

$$= m \frac{d v_x}{dt} + q \left( \frac{d x}{d t} \frac{\partial A_x}{\partial x} + \frac{d y}{d t} \frac{\partial A_x}{\partial y} + \frac{d z}{d t} \frac{\partial A_x}{\partial z} + \frac{\partial A_x}{\partial t} \right)$$

$$= m \frac{d v_x}{dt} + q \left( v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_x}{\partial z} + \frac{\partial A_x}{\partial t} \right),$$

$$\frac{\partial L}{\partial x} = -q \frac{\partial V}{\partial x} + q \left( v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_y}{\partial x} + v_z \frac{\partial A_z}{\partial x} \right)$$

$$m \frac{d v_x}{dt} = -q \left( \frac{\partial V}{\partial x} + \frac{\partial A_x}{\partial t} \right) + q \left( v_y \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) + v_z \left( \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \right)$$

# Lagrangian

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$$m \frac{dv_x}{dt} = -q \left( \frac{\partial V}{\partial x} + \frac{\partial A_x}{\partial t} \right) + q \left( v_y \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) + v_z \left( \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \right)$$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \quad \text{and} \quad \vec{B} = \nabla \times \vec{A}$$

$$m \frac{dv_x}{dt} = q \left( E_x + (\vec{v} \times \vec{B})_x \right)$$

$$L = \frac{1}{2} m v^2 - qV(\vec{r}, t) + q\vec{v} \cdot \vec{A}(\vec{r}, t)$$

conjugate variable:  $p_x = \frac{\partial L}{\partial v_x} = m v_x + q A_x$

kinetic momentum  $v_x = \frac{1}{m} (p_x - q A_x)$  field momentum (inductance)

# Hamiltonian

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$$v_x = \frac{1}{m}(p_x - qA_x)$$

$$L = \frac{1}{2}mv^2 - qV(\vec{r}, t) + q\vec{v} \cdot \vec{A}(\vec{r}, t)$$

Legendre transformation  $H = \vec{v} \cdot \vec{p} - L$

Classical result  $\rightarrow H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + qV$

$$\vec{p} \rightarrow -i\hbar\nabla$$

Schrödinger equation  $i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m}(-i\hbar\nabla - q\vec{A})^2 \psi + qV\psi$

# Landau Levels

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free particles in a magnetic field

$$\frac{1}{2m} \left( -i\hbar\nabla - q\vec{A} \right)^2 \psi(\vec{r}) = E\psi(\vec{r}). \quad V=0$$

Landau gauge  $\vec{A} = B_z x \hat{y}.$

$$\vec{B} = \nabla \times \vec{A} = \nabla \times B_z x \hat{y} = \left( \frac{dA_z}{dy} - \frac{dA_y}{dz} \right) \hat{x} + \left( \frac{dA_x}{dz} - \frac{dA_z}{dx} \right) \hat{y} + \left( \frac{dA_y}{dx} - \frac{dA_x}{dy} \right) \hat{z}.$$

$$\vec{B} = B_z \hat{z}.$$



# Landau Levels

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free particles in a magnetic field

$$\frac{1}{2m} \left( -i\hbar\nabla - q\vec{A} \right)^2 \psi(\vec{r}) = E\psi(\vec{r}). \quad V=0$$

Landau gauge  $\vec{A} = B_z x \hat{y}$ .

$$\left( -i\hbar\nabla - q\vec{A} \right)^2 = \left( -i\hbar\nabla - qB_z x \hat{y} \right) \cdot \left( -i\hbar\nabla - qB_z x \hat{y} \right)$$

$$-i\hbar\nabla \cdot \left( -qB_z x \hat{y} \right) = -i\hbar \left( \frac{d}{dx} \hat{x} + \frac{d}{dy} \hat{y} + \frac{d}{dz} \hat{z} \right) \cdot \left( -qB_z x \hat{y} \right) = i\hbar q B_z x \frac{d}{dy}$$

$$\frac{1}{2m} \left( -\hbar^2 \nabla^2 + i2\hbar q B_z x \frac{d}{dy} + q^2 B_z^2 x^2 \right) \psi = E\psi.$$

The solution has the form

$$\psi = e^{ik_y y} e^{ik_z z} \phi(x).$$

# Landau Levels

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$$\frac{1}{2m} \left( -\hbar^2 \nabla^2 + i2\hbar q B_z x \frac{d}{dy} + q^2 B_z^2 x^2 \right) \psi = E\psi.$$

The solution has the form

$$\psi = e^{ik_y y} e^{ik_z z} \phi(x).$$

Substitute this into the equation

$$\frac{1}{2m} \left( -\hbar^2 \frac{\partial^2}{\partial x^2} + \hbar^2 k_z^2 + \hbar^2 k_y^2 - 2\hbar q B_z k_y x + q^2 B_z^2 x^2 \right) \phi(x) = E\phi(x).$$

# Landau Levels

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The equation for  $\phi(x)$  is

$$\frac{1}{2m} \left( -\hbar^2 \frac{\partial^2}{\partial x^2} + \hbar^2 k_z^2 + \underbrace{\hbar^2 k_y^2 - 2\hbar q B_z k_y x + q^2 B_z^2 x^2}_{(\hbar k_y - q B_z x)^2} \right) \phi(x) = E \phi(x).$$

$$\frac{1}{2m} \left( -\hbar^2 \frac{\partial^2}{\partial x^2} + q^2 B_z^2 \left( x - \frac{\hbar k_y}{q B_z} \right)^2 \right) \phi(x) = \left( E - \frac{\hbar^2 k_z^2}{2m} \right) \phi(x).$$

This is the equation for a harmonic oscillator

$$\left( \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{K}{2} (x - x_0)^2 \right) \phi(x) = E' \phi(x).$$

# Landau Levels

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$$\frac{1}{2m} \left( -\hbar^2 \frac{\partial^2}{\partial x^2} + q^2 B_z^2 (x - x_0)^2 \right) \phi(x) = \left( E - \frac{\hbar^2 k_z^2}{2m} \right) \phi(x). \quad x_0 = \frac{\hbar k_y}{q B_z}$$
$$\left( \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{K}{2} (x - x_0)^2 \right) \phi(x) = E' \phi(x).$$

This is the equation for a harmonic oscillator

$$\frac{K}{2} \Leftrightarrow \frac{q^2 B_z^2}{2m}$$

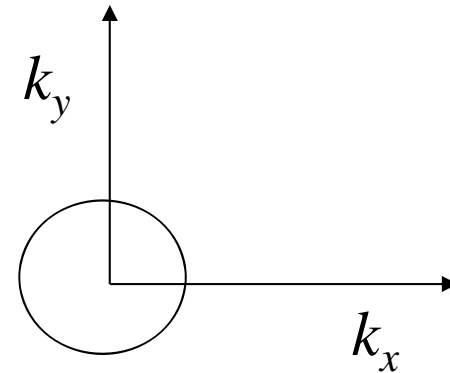
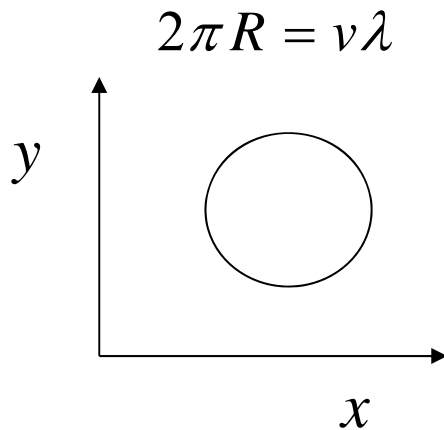
$$\omega_c = \sqrt{\frac{K}{m}} \Leftrightarrow \frac{q B_z}{m}$$

$$E_{k_z, \nu} = \frac{\hbar^2 k_z^2}{2m} + \hbar \omega_c \left( \nu + \frac{1}{2} \right) \quad \omega_c = \frac{q B_z}{m}$$

# Charged particle in a magnetic field

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Bohr - Sommerfeld quantization



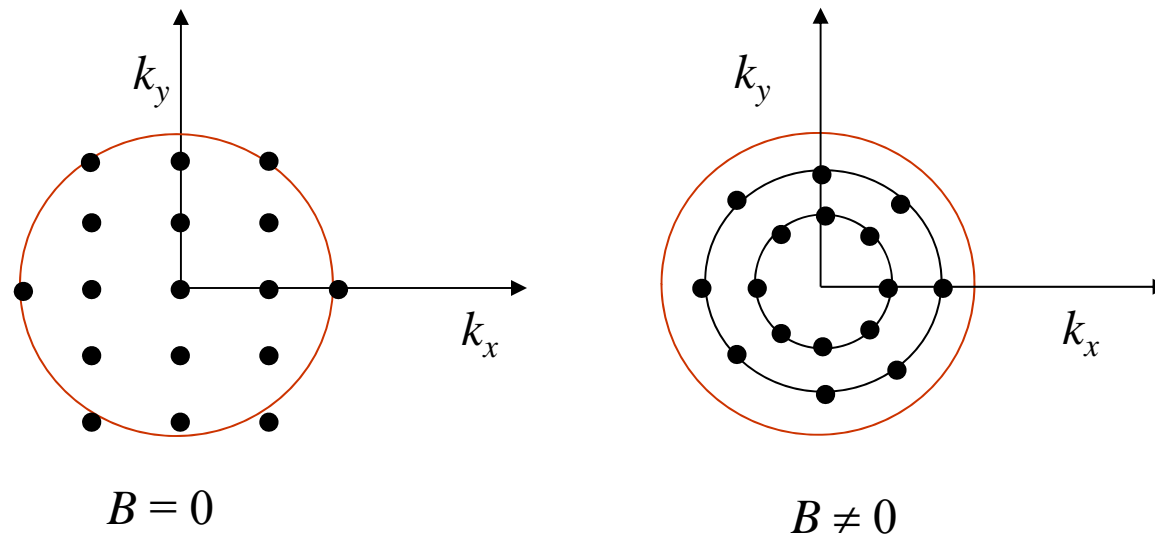
Circular motion is harmonic motion. Harmonic motion is quantized.

$$E_\nu = \hbar\omega_c \left(\nu + \frac{1}{2}\right) = \frac{\hbar^2}{2m} (k_x^2 + k_y^2)$$

$\nu$  labels the Landau level

# Landau levels

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The number of solutions is conserved

$$\psi = e^{ik_y y} \phi(x - x_0). \quad x_0 = \frac{\hbar k_y}{qB_z}$$

In 2-D, the  $k$ -volume per  $k$  state is:  $\left(\frac{2\pi}{L}\right)^2$