Free Electrons in a Magnetic Field

Charged particle in a magnetic field

$$\vec{F} = -e\vec{v} \times \vec{B} = ma$$

$$evB_z = \frac{mv^2}{R}$$

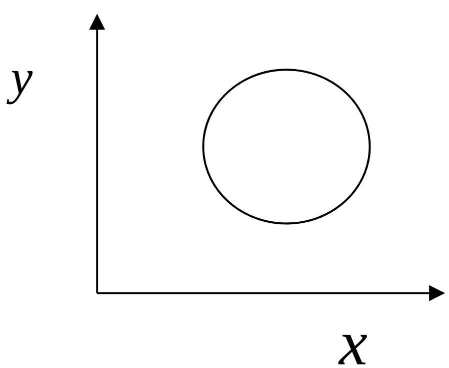
solve for velocity

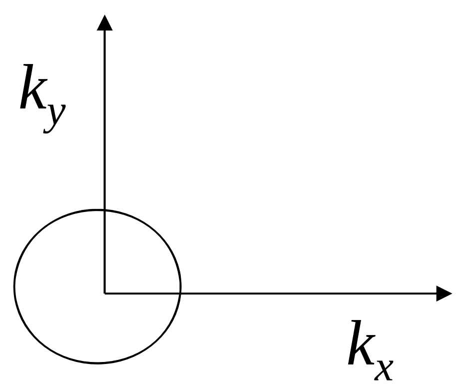
$$v = \frac{eB_z R}{m}$$

$$v = \frac{2\pi R}{T} = \omega_c R$$

$$\omega_c = \frac{eB_z}{m}$$

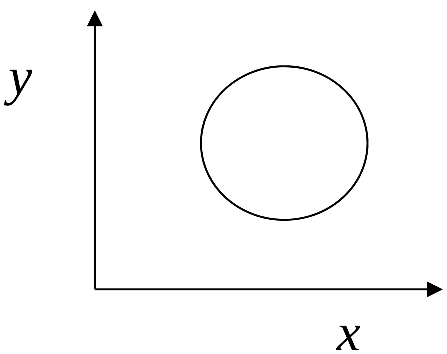
Magnetron



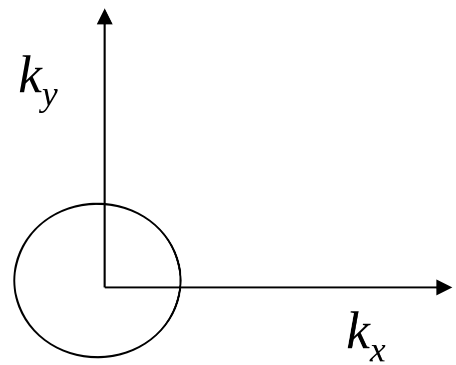


Why don't metals in a B field radiate?

Magnetron



$$\omega_c = \frac{eB_z}{m}$$



There are no lower lying states to fall into. We need a quantum description.

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

Magnetic force depends on the velocity, not on the position.

Electrons in a magnetic field

Lorentz force law $\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$

Euler Lagrange equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$

Lagrangian: $L = \tfrac{1}{2} m v^2 - qV\left(\vec{r}, t\right) + q\vec{v} \cdot \vec{A}\left(\vec{r}, t\right)$

Kittel: Appendix G

http://lamp.tu-graz.ac.at/~hadley/ss2/IQHE/cpimf.php

Lagrangian

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)-\frac{\partial L}{\partial x_i}=0 \qquad L=\tfrac{1}{2}mv^2-qV\left(\vec{r},t\right)+q\vec{v}\cdot\vec{A}\left(\vec{r},t\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_x}\right)=\frac{d}{dt}\left(mv_x+qA_x\right)=m\frac{dv_x}{dt}+q\frac{dA_x}{dt}$$

$$=m\frac{dv_x}{dt}+q\left(\frac{dx}{dt}\frac{\partial A_x}{\partial x}+\frac{dy}{dt}\frac{\partial A_x}{\partial y}+\frac{dz}{dt}\frac{\partial A_x}{\partial z}+\frac{\partial A_x}{\partial t}\right)$$

$$=m\frac{dv_x}{dt}+q\left(v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_x}{\partial y}+v_z\frac{\partial A_x}{\partial z}+\frac{\partial A_x}{\partial t}\right),$$

$$\frac{\partial L}{\partial x}=-q\frac{\partial V}{\partial x}+q\left(v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_y}{\partial x}+v_z\frac{\partial A_z}{\partial x}\right)$$

$$m\frac{dv_x}{dt}=-q\left(\frac{\partial V}{\partial x}+\frac{\partial A_x}{\partial t}\right)+q\left(v_y\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)+v_z\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right)\right)$$

Lagrangian

$$m\frac{dv_x}{dt} = -q\left(\frac{\partial V}{\partial x} + \frac{\partial A_x}{\partial t}\right) + q\left(v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + v_z\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\right)$$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \qquad \text{and} \qquad \vec{B} = \nabla \times \vec{A}$$

$$m\frac{dv_x}{dt} = q\left(E_x + \left(\vec{v}\times\vec{B}\right)_x\right)$$

$$L = \tfrac{1}{2}mv^2 - qV\left(\vec{r},t\right) + q\vec{v}\cdot\vec{A}\left(\vec{r},t\right)$$

conjugate variable: $$p_x = \frac{\partial L}{\partial v_x} = mv_x + qA_x$$

kinetic momentum

field momentum (inductance)

$$v_x = \frac{1}{m}\left(p_x - qA_x\right)$$

Hamiltonian

$$v_x = \frac{1}{m}\left(p_x - qA_x\right)$$

$$L = \tfrac{1}{2} m v^2 - qV\left(\vec{r},t\right) + q\vec{v}\cdot\vec{A}\left(\vec{r},t\right)$$

Legendre transformation $H = \vec{v}\cdot\vec{p} - L$

$$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + qV$$

Classical result

$$\vec{p} \rightarrow -i\hbar\nabla$$

Schrödinger equation $$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(-i\hbar\nabla - q\vec{A}\right)^2\psi + qV\psi$$

Landau Levels

free particles in a magnetic field

$$\frac{1}{2m}\left(-i\hbar\nabla - q\vec{A}\right)^2 \psi(\vec{r}) = E\psi(\vec{r}). \qquad V = 0$$

Landau gauge $\qquad \vec{A} = B_z x\hat{y}.$

$$\vec{B} = \nabla \times \vec{A} = \nabla \times B_z x\hat{y} = \left(\frac{dA_z}{dy} - \frac{dA_y}{dz}\right)\hat{x} + \left(\frac{dA_x}{dz} - \frac{dA_z}{dx}\right)\hat{y} + \left(\frac{dA_y}{dx} - \frac{dA_x}{dy}\right)\hat{z}.$$

$$\vec{B} = B_z\hat{z}.$$

Landau Levels

free particles in a magnetic field

$$\frac{1}{2m}\left(-i\hbar\nabla - q\vec{A}\right)^2 \psi(\vec{r}) = E\psi(\vec{r}). \qquad V = 0$$

Landau gauge $\vec{A} = B_z x\hat{y}.$

$$\left(-i\hbar\nabla - q\vec{A}\right)^2 = \left(-i\hbar\nabla - qB_z x\hat{y}\right)\cdot\left(-i\hbar\nabla - qB_z x\hat{y}\right)$$

$$-i\hbar\nabla\cdot\left(-qB_z x\hat{y}\right) = -i\hbar\left(\frac{d}{dx}\hat{x} + \frac{d}{dy}\hat{y} + \frac{d}{dz}\hat{z}\right)\cdot\left(-qB_z x\hat{y}\right) = i\hbar qB_z x\frac{d}{dy}$$

$$\frac{1}{2m}\left(-\hbar^2\nabla^2 + i2\hbar qB_z x\frac{d}{dy} + q^2 B_z^2 x^2\right)\psi = E\psi.$$

The solution has the form

$$\psi = e^{ik_y y} e^{ik_z z}\phi(x).$$

Landau Levels

$$\frac{1}{2m}\left(-\hbar^2\nabla^2 + i2\hbar qB_z x\frac{d}{dy} + q^2B_z^2x^2\right)\psi = E\psi.$$

The solution has the form

$$\psi = e^{ik_y y}e^{ik_z z}\phi(x).$$

Substitute this into the equation

$$\frac{1}{2m}\left(-\hbar^2\frac{\partial^2}{\partial x^2} + \hbar^2k_z^2 + \hbar^2k_y^2 - 2\hbar qB_z k_y x + q^2B_z^2x^2\right)\phi(x) = E\phi(x).$$

Landau Levels

The equation for $\phi(x)$ is

$$\frac{1}{2m}\left(-\hbar^2\frac{\partial^2}{\partial x^2}+\hbar^2k_z^2+\underbrace{\hbar^2k_y^2-2\hbar qB_zk_yx+q^2B_z^2x^2}_{\left(\hbar k_y-qB_zx\right)^2}\right)\phi(x)=E\phi(x).$$

$$\frac{1}{2m}\left(-\hbar^2\frac{\partial^2}{\partial x^2}+q^2B_z^2\left(x-\frac{\hbar k_y}{qB_z}\right)^2\right)\phi(x)=\left(E-\frac{\hbar^2k_z^2}{2m}\right)\phi(x).$$

This is the equation for a harmonic oscillator

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{K}{2}\left(x-x_0\right)^2\right)\phi(x)=E'\phi(x).$$

Landau Levels

$$\frac{1}{2m}\left(-\hbar^2\frac{\partial^2}{\partial x^2}+q^2B_z^2\left(x-x_0\right)^2\right)\phi(x)=\left(E-\frac{\hbar^2k_z^2}{2m}\right)\phi(x). \quad x_0=\frac{\hbar k_y}{qB_z}$$

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{K}{2}\left(x-x_0\right)^2\right)\phi(x)=E'\phi(x).$$

This is the equation for a harmonic oscillator

$$\frac{K}{2}\Leftrightarrow\frac{q^2B_z^2}{2m}$$

$$\omega_c=\sqrt{\frac{K}{m}}\Leftrightarrow\frac{qB_z}{m}$$

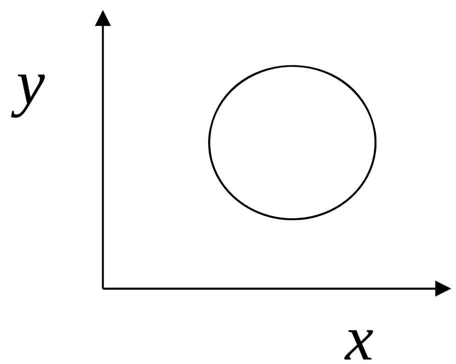
$$E_{k_z,v}=\frac{\hbar^2k_z^2}{2m}+\hbar\omega_c\left(v+\tfrac{1}{2}\right)$$

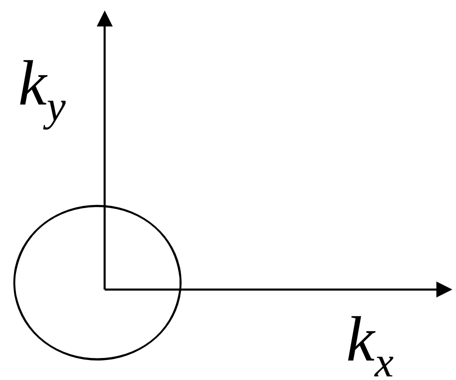
$$\omega_c=\frac{qB_z}{m}$$

Charged particle in a magnetic field

Bohr - Sommerfeld quantization

$$2\pi R = \nu\lambda$$



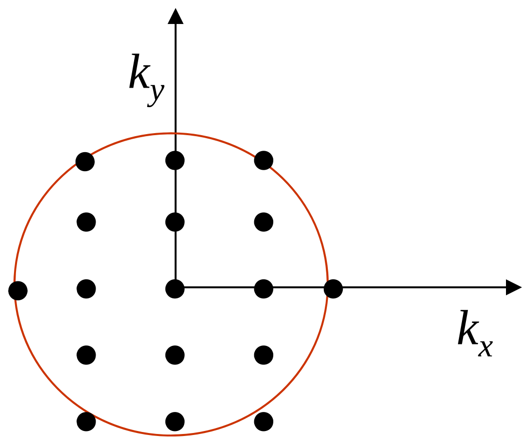


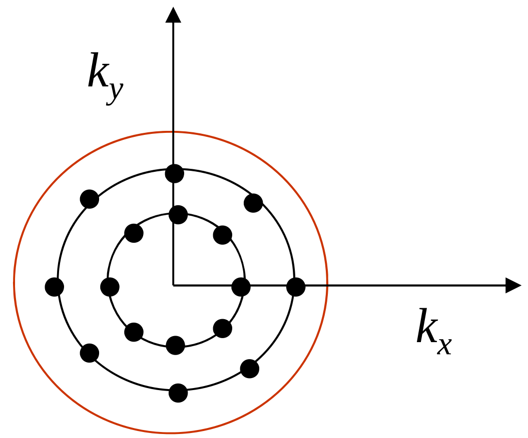
Circular motion is harmonic motion. Harmonic motion is quantized.

$$E_\nu = \hbar\omega_c\left(\nu + \tfrac{1}{2}\right) = \frac{\hbar^2}{2m}\left(k_x^2 + k_y^2\right)$$

ν labels the Landau level

Landau levels





The number of solutions is conserved

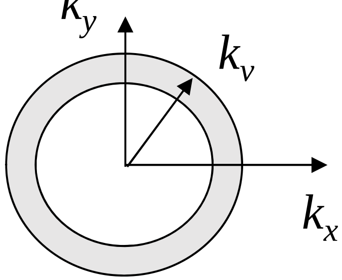
$$\psi = e^{ik_y y}\phi(x - x_0). \qquad x_0 = \frac{\hbar k_y}{qB_z}$$

In 2-D, the k-volume per k state is: $\left(\frac{2\pi}{L}\right)^2$

Density of states 2D

$$E_\nu = \hbar\omega_c\left(\nu + \tfrac{1}{2}\right)$$

The number of states between ring *ν*-1 and ring *ν* is



$$\frac{\pi\left(k_\nu^2 - k_{\nu-1}^2\right)}{\left(\dfrac{2\pi}{L}\right)^2}$$

$$\frac{\hbar^2 k_\nu^2}{2m} = \hbar\omega_c\left(\nu + \tfrac{1}{2}\right)$$

$$k_\nu^2 - k_{\nu-1}^2 = \frac{2m\omega_c}{\hbar}\left[\left(\nu + \tfrac{1}{2}\right) - \left(\nu - 1 + \tfrac{1}{2}\right)\right] = \frac{2m\omega_c}{\hbar}$$

The number of states between ring *ν*-1 and ring *ν* is $\dfrac{m\omega_c}{2\pi\hbar}L^2$

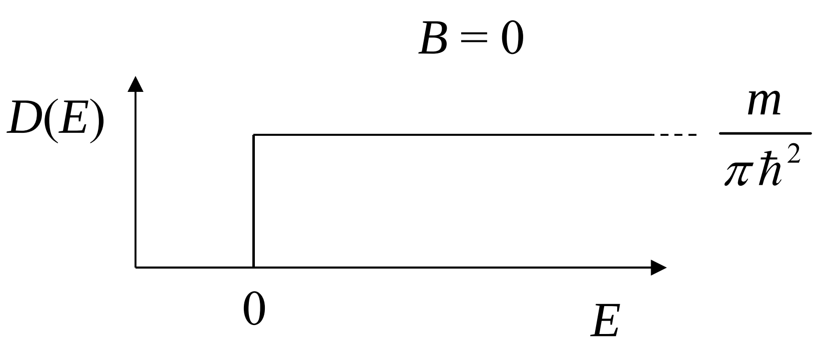
The density of states per spin is $\dfrac{m\omega_c}{2\pi\hbar}$

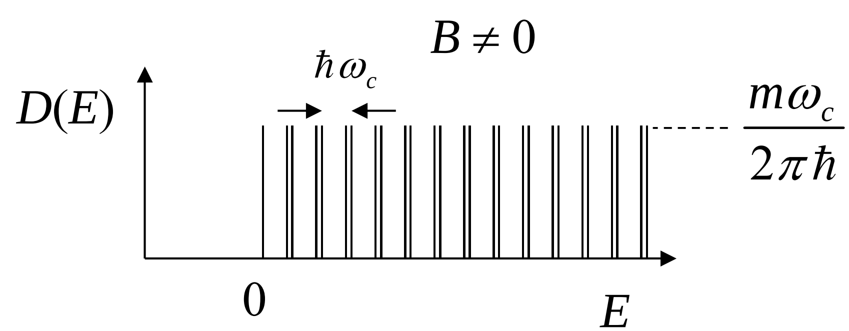
Spin

In a magnetic field, there is a shift of the energy of the electrons because of their spin.

$$E = -\vec{\mu}\cdot\vec{B} = \pm\frac{g}{2}\mu_B B$$

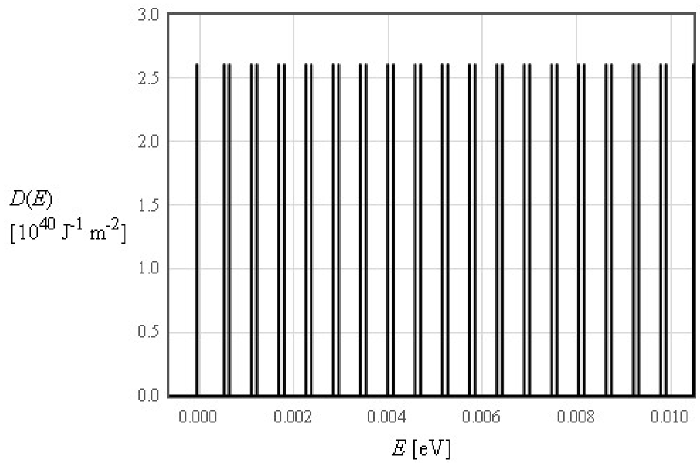
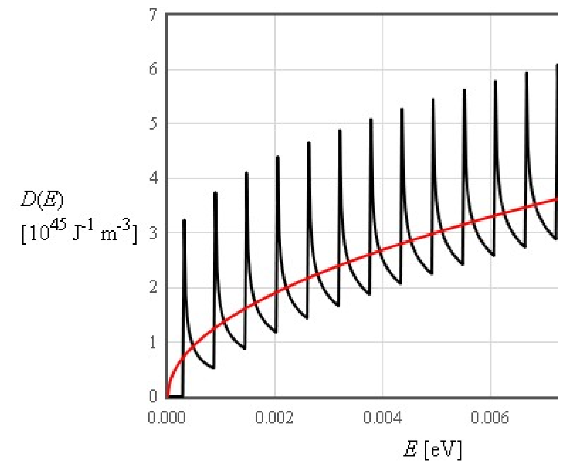
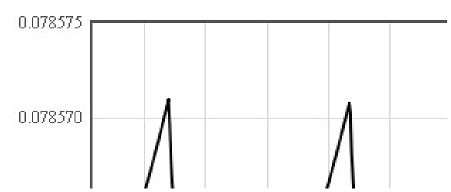
Bohr magneton $\mu_B = \dfrac{e\hbar}{2m_e}$ $\quad$ g-factor $g \approx 2$ $\quad$ $\hbar\omega_c = \dfrac{\hbar e B}{m} = 2\mu_B B$





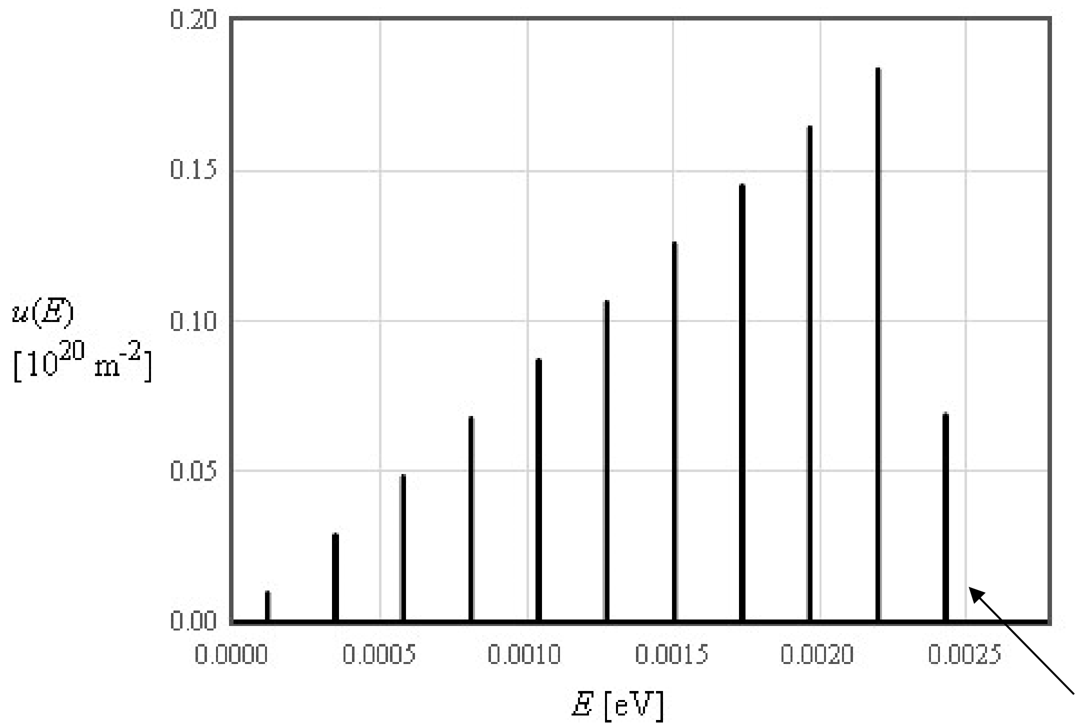
$$D(E) = \frac{m\omega_c}{2\pi\hbar}\sum_{\nu=0}^{\infty}\delta\left(E-\hbar\omega_c\left(\nu+\tfrac{1}{2}-\tfrac{g}{4}\right)\right)+\delta\left(E-\hbar\omega_c\left(\nu+\tfrac{1}{2}+\tfrac{g}{4}\right)\right)$$

ization of the Schrödinger equation for free electrons a magnetic field in 2 and 3 dimensions.

	2-D Schrödinger equation $i\hbar\frac{d\psi}{dt}=\frac{1}{2m}\left(-i\hbar\nabla-\lvert e\rvert\vec{A}\right)^2\psi$	3-D Schrödinger equation $i\hbar\frac{d\psi}{dt}=\frac{1}{2m}\left(-i\hbar\nabla-\lvert e\rvert\vec{A}\right)^2\psi$
Eigenfunction solutions	$\psi=g_v(x)\exp\left(ik_y y\right)$ $g_v(x)$ is a harmonic oscillator wavefunction	$\psi=g_v(x)\exp\left(ik_y y\right)\exp\left(ik_z z\right)$ $g_v(x)$ is a harmonic oscillator wavefun
Energy eigenvalues	$E=\hbar\omega_c\left(v+\tfrac{1}{2}\right)$ J $v=0,1,2,\cdots$ $\quad\omega_c=\frac{\lvert eB_z\rvert}{m}$	$E=\frac{\hbar^2k_z^2}{2m}+\hbar\omega_c\left(v+\tfrac{1}{2}\right)$ J $v=0,1,2,\cdots$ $\quad\omega_c=\frac{\lvert eB_z\rvert}{m}$
Density of states	$D(E)=\frac{m\omega_c}{2\pi\hbar}\sum_{v=0}^{\infty}\delta\left(E-\hbar\omega_c\left(v+\tfrac{1}{2}\right)-\frac{g\mu_B}{2}B\right)+\delta\left(E-\hbar\omega_c\left(v+\tfrac{1}{2}\right)+\frac{g\mu_B}{2}B\right)$ $\mathrm{J^{-1}m^{-2}}$  Calculate DoS	$D(E)=\frac{(2m)^{3/2}\,\omega_c}{4\pi^2\hbar^2}\sum_{v=0}^{\infty}\frac{H\left(E-\hbar\omega_c\left(v+\frac{1}{2}\right)\right)}{\sqrt{E-\hbar\omega_c\left(v+\frac{1}{2}\right)}}$  Calculate DoS
	$E_F=\hbar\omega_c\left(\mathrm{Int}\left(\frac{\pi\hbar n}{m\omega_c}\right)+\frac{1}{2}\right)$ 	

Energy spectral density 2d

At $T = 0$

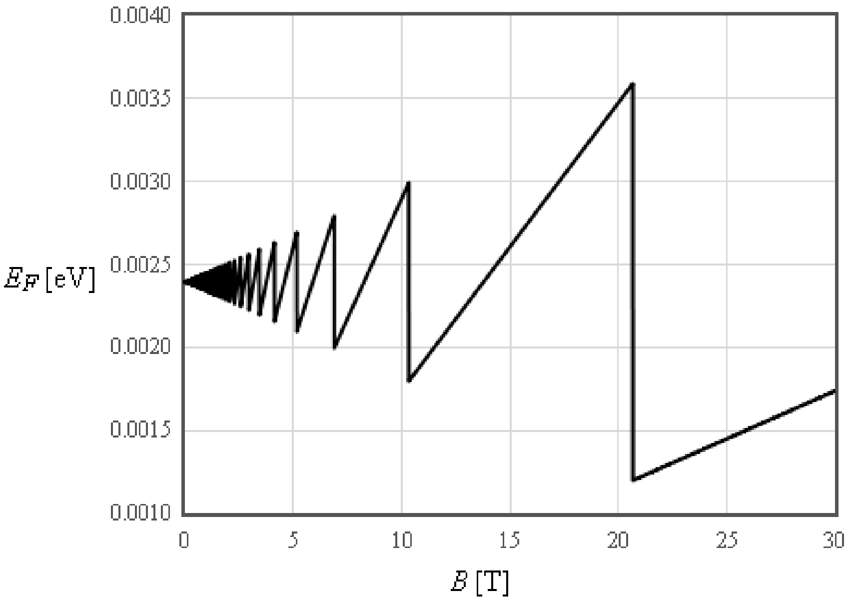


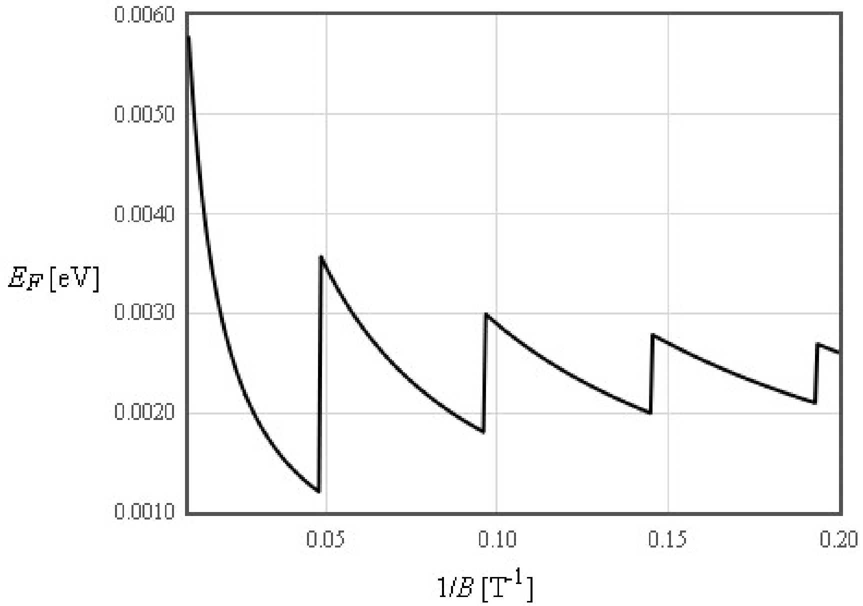
unfilled Landau level

analog to the Planck radiation law

Fermi energy 2d

$$n = \int_{-\infty}^{E_F} D(E)dE$$





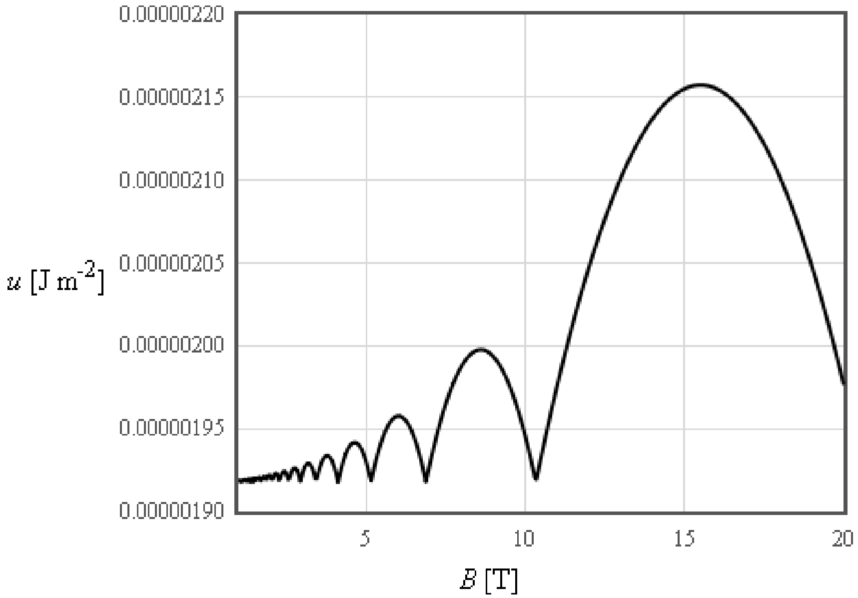
When there is only one Landau level, the Fermi energy rises linearly with field.

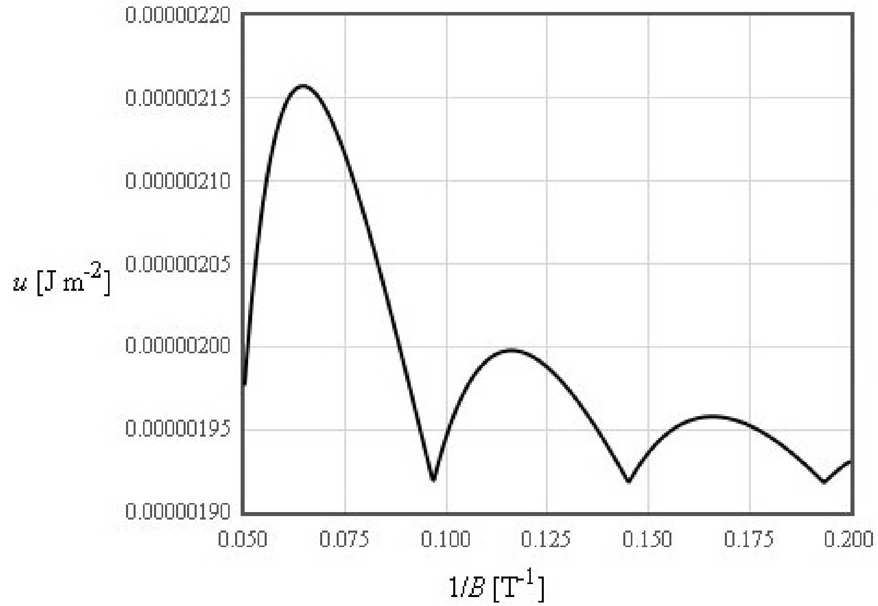
Periodic in $1/B$

Large field limit $\longrightarrow$ $E_F = \dfrac{\hbar\omega_c}{2} = \dfrac{\hbar eB}{2m}$

Internal energy 2d

At $T = 0$ $\quad u = \int_{-\infty}^{E_F} E D(E) dE$



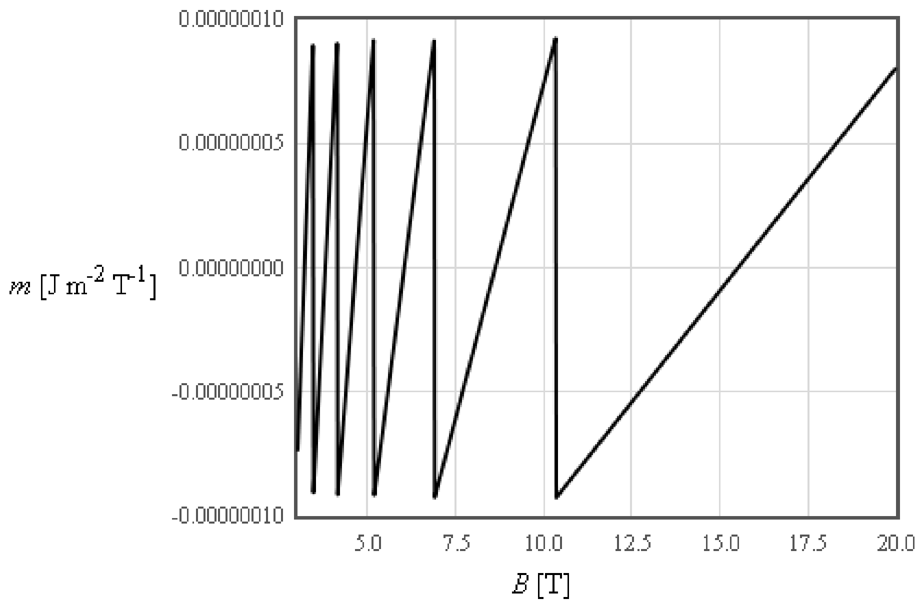


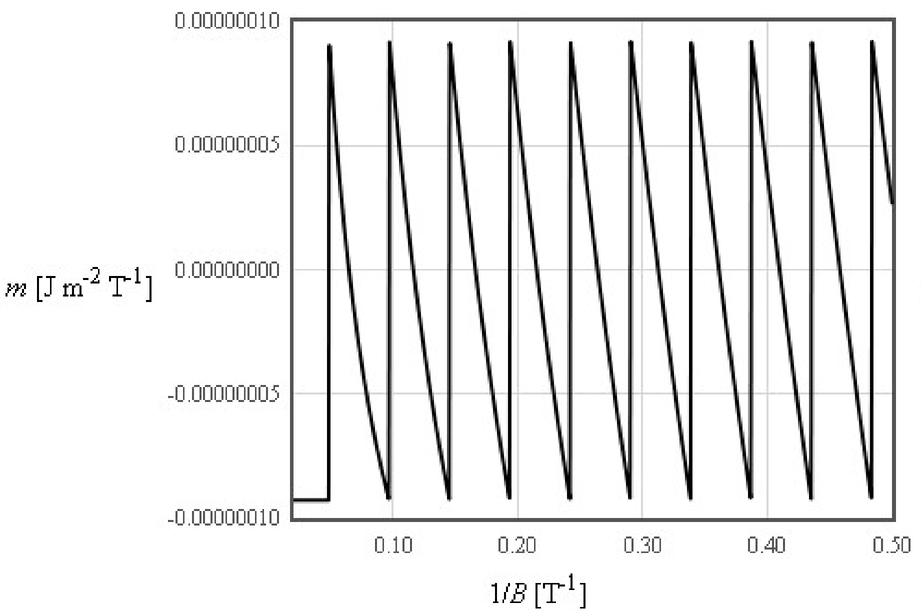
$$u = n\frac{\hbar\omega_c}{2} = n\frac{\hbar eB}{2m}$$

Large field limit

Magnetization 2d

At $T = 0$



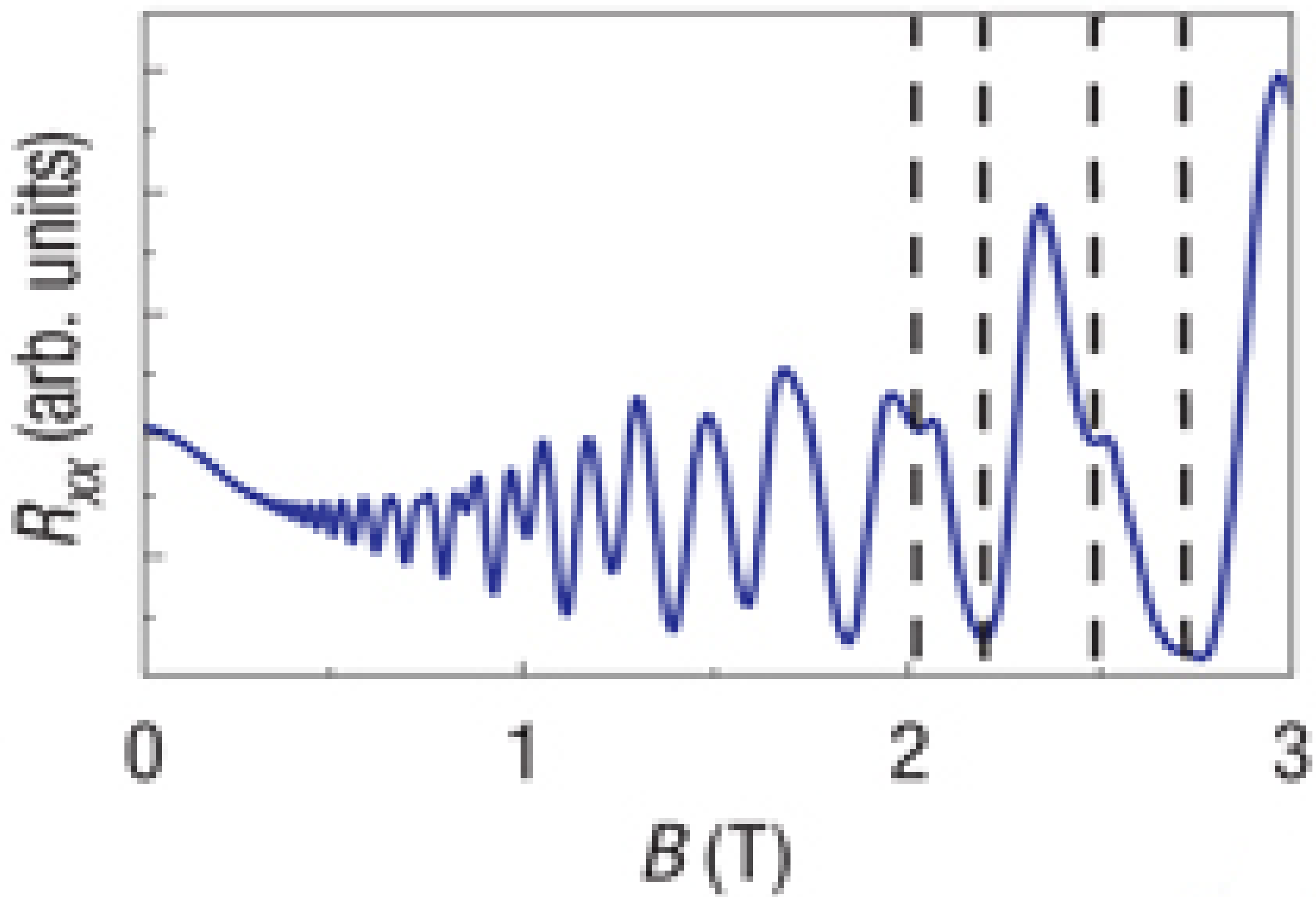


$$m = -\frac{du}{dB} = -n\frac{\hbar e}{2m}$$

Large field limit

de Haas - van Alphen oscillations

Shubnikov-De Haas oscillations



Scattering at the Fermi surface

At room temperature, phonon energies are much less than the Fermi energy. The energy of electrons hardly changes as they scatter from phonons. Electrons scatter from a point close to the Fermi surface to another point close to the Fermi surface.

Changing the magnetic field changes the number of states at the Fermi energy.

There are oscillations in the electrical conductivity as a function of magnetic field.