Superconductivity

BCS theory (1957)

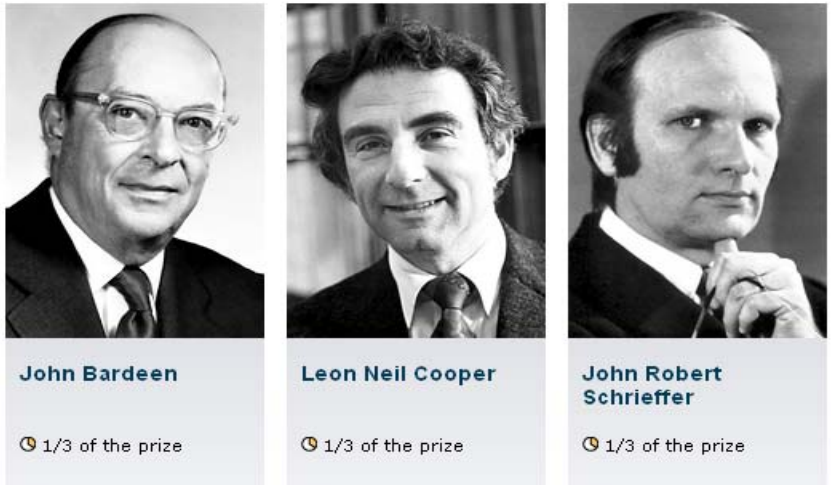
Electrons form Cooper pairs

Electrons condense into a coherent state. Similar to:

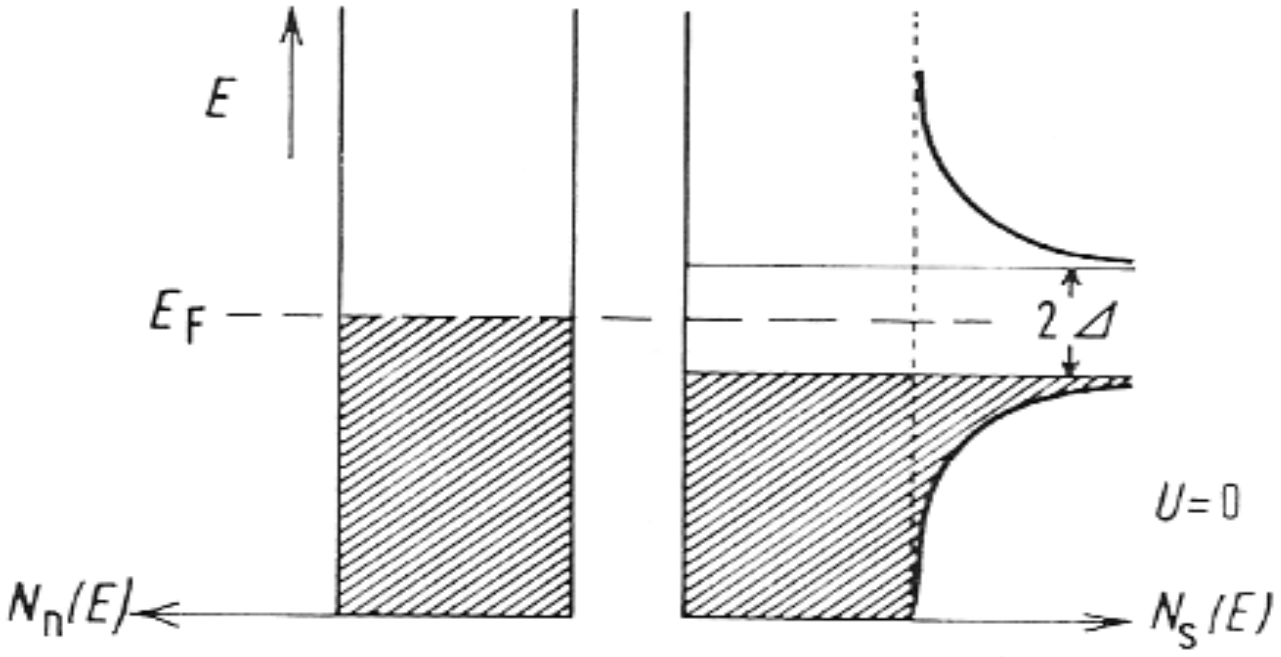
- Superfluidity
- Bose-Einstein condensates
- Lasers

Pauli exclusion: the sign of the wavefunction changes when two electrons are exchanged.

1972



Density of states



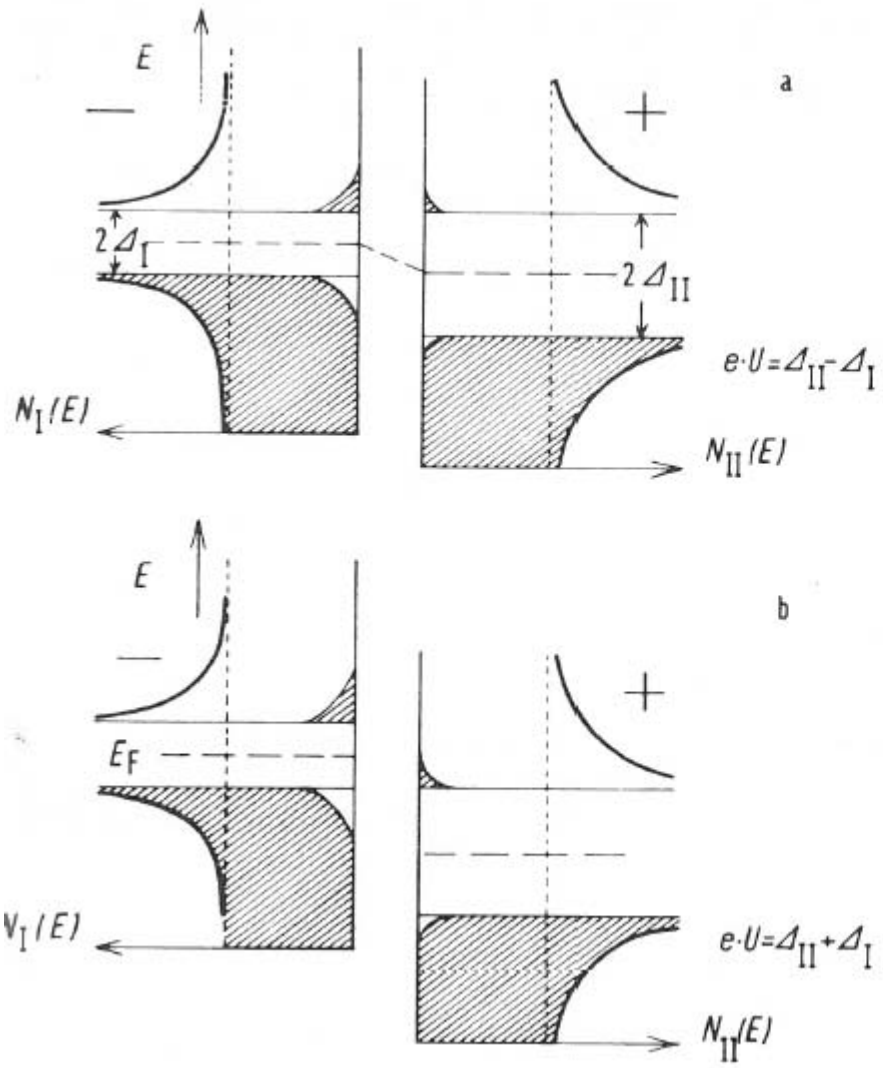
Condensate at E_F

Build wave packets out of states near E_F - Cooper pairs

exchange electrons $\Psi \rightarrow -\Psi$ exchange CP $\Psi \rightarrow \Psi$

no states within Δ of E_F

Tunneling spectroscopy



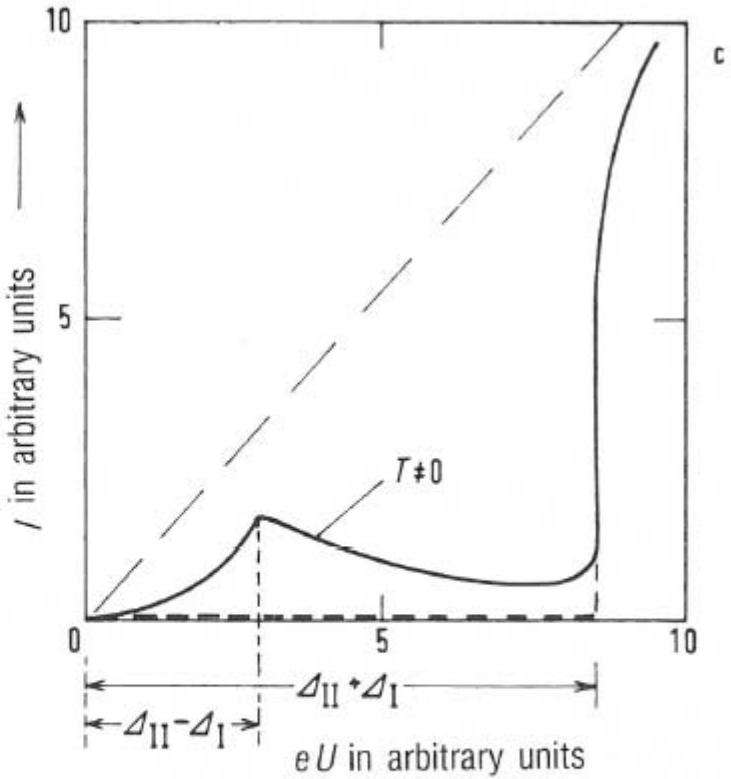


Fig. 31. Tunneling current between two superconductors $0 < T < T_c$.

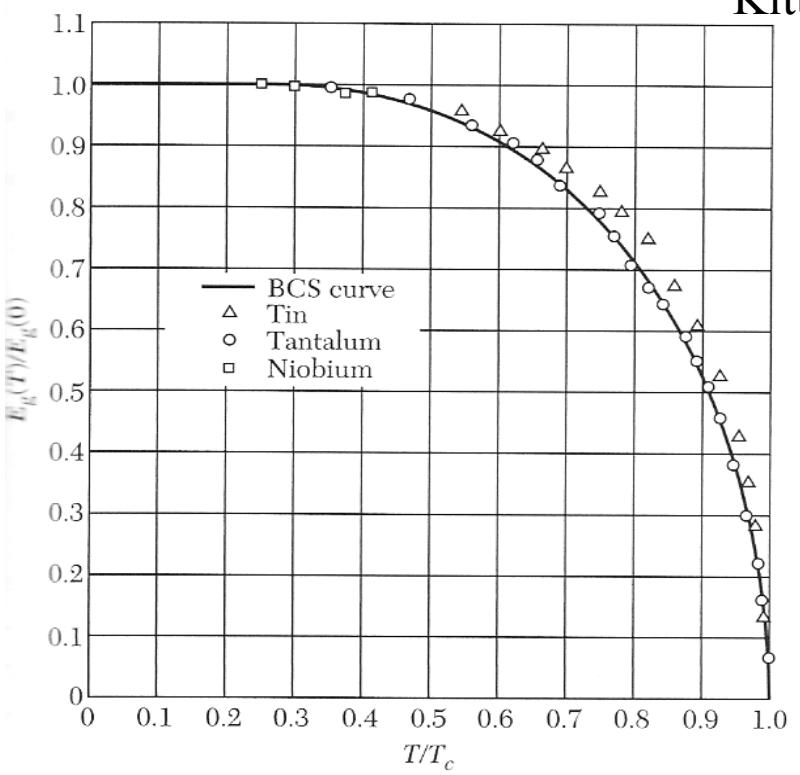
Buckel - Superconductivity

BCS results

$$\frac{\Delta(0)}{k_B T} = 1.76$$

Al	1.7
Cd	1.6
In	1.8
Hg	2.3
Nb	1.9
Pb	2.1
Sn	1.7

Kittel



$$\left.\frac{c_s - c_n}{c_n}\right|_{T=T_c} = 1.43$$

Al	1.4
Cd	1.4
In	1.7
Hg	2.4
Nb	1.9
Pb	2.7
Sn	1.6

Superconductivity

Perfect diamagnetism

Electrons condense into a single quantum state - low entropy.

Electron decrease their energy by Δ but loose their entropy.

Jump in the specific heat like a 2nd order phase transition, not a structural transition

Superconductors are good electrical conductors but poor thermal conductors, electrons no longer conduct heat

There is a dramatic decrease of acoustic attenuation at the phase transition, no electron-phonon scattering

Dissipationless currents - quantum effect

Probability current

Schrödinger equation for a charged particle in an electric and magnetic field is

$$i\hbar\frac{\partial\psi}{\partial t}=\frac{1}{2m}(-i\hbar\nabla-qA)^2\psi+V\psi$$

write out the $(-i\hbar\nabla-qA)^2\psi$ term

$$i\hbar\frac{\partial\psi}{\partial t}=\frac{1}{2m}\left(-\hbar^2\nabla^2+i\hbar qA\nabla+ihq\nabla A+q^2A^2\right)\psi+V\psi$$

write the wave function in polar form

$$\psi=|\psi|e^{i\theta}$$

$$\nabla\psi=\nabla|\psi|e^{i\theta}+i\nabla\theta|\psi|e^{i\theta}$$

$$\nabla^2\psi=\nabla^2|\psi|e^{i\theta}+2i\nabla\theta\nabla|\psi|e^{i\theta}+i\nabla^2\theta|\psi|e^{i\theta}-(\nabla\theta)^2|\psi|e^{i\theta}$$

Probability current

Schrödinger equation becomes:

$$i\hbar\frac{\partial|\psi|}{\partial t}-\hbar|\psi|\frac{\partial\theta}{\partial t}=\frac{1}{2m}\Big[-\hbar^2\left(\nabla^2|\psi|+2i\nabla\theta\nabla|\psi|+i\nabla^2\theta|\psi|-(\nabla\theta)^2|\psi|\right)$$
$$+i\hbar qA\left(\nabla|\psi|+i\nabla\theta|\psi|\right)+ihq\nabla A|\psi|+q^2A^2|\psi|\Big]+V|\psi|$$

Real part:

$$-\hbar|\psi|\frac{\partial\theta}{\partial t}=\frac{-\hbar^2}{2m}\left(\nabla^2-\left(\nabla\theta-\frac{q}{\hbar}\vec{A}\right)^2\right)|\psi|+V|\psi|$$

Imaginary part:

$$\hbar\frac{\partial|\psi|}{\partial t}=\frac{1}{2m}\Big[-\hbar^2\left(2\nabla\theta\nabla|\psi|+i\nabla^2\theta|\psi|-(\nabla\theta)^2|\psi|\right)+2\hbar qA\nabla|\psi|+\hbar q|\psi|\nabla A\Big]$$

Probability current

Imaginary part:

$$\hbar\frac{\partial|\psi|}{\partial t}=\frac{1}{2m}\left[-\hbar^2\left(2\nabla\theta\nabla|\psi|+i\nabla^2\theta|\psi|-(\nabla\theta)^2|\psi|\right)+2\hbar qA\nabla|\psi|+\hbar q|\psi|\nabla A\right]$$

Multiply by $2|\psi|$ and rearrange $\quad 2|\psi|\frac{\partial|\psi|}{\partial t}=\frac{\partial}{\partial t}|\psi|^2$

$$\frac{\partial}{\partial t}|\psi|^2+\nabla\cdot\left[\frac{\hbar}{m}|\psi|^2\left(\nabla\theta-\frac{q}{\hbar}\vec{A}\right)\right]=0$$

This is a continuity equation for probability

$$\frac{\partial P}{\partial t}+\nabla\cdot\vec{S}=0$$

The probability current: $\quad \vec{S}=\frac{\hbar}{m}|\psi|^2\left(\nabla\theta-\frac{q}{\hbar}\vec{A}\right)$

Probability current / supercurrent

The probability current: $\vec{S} = \frac{\hbar}{m}|\psi|^2\left(\nabla\theta - \frac{q}{\hbar}\vec{A}\right)$

This result holds for all charged particles in a magnetic field.

In superconductivity the particles are Cooper pairs $q = -2e$, $m = 2m_e$, $|\psi|^2 = n_{cp}$.

All superconducting electrons are in the same state so

$$\vec{j} = -2e\vec{S}$$

$$\boxed{\vec{j} = \frac{-e\hbar n_{cp}}{m_e}\left(\nabla\theta + \frac{2e}{\hbar}\vec{A}\right)}$$

London gauge $\nabla\theta = 0$

$$\vec{j} = \frac{-2n_{cp}e^2}{m_e}\vec{A} = \frac{-n_s e^2}{m_e}\vec{A} \qquad n_s = 2n_{cp}$$

1st London equation

Heinz & Fritz

$$\vec{j} = \frac{-n_s e^2}{m_e}\vec{A}$$

$$\frac{d\vec{j}}{dt} = \frac{-n_s e^2}{m_e}\frac{d\vec{A}}{dt} = \frac{n_s e^2}{m_e}\vec{E} \qquad \frac{d\vec{A}}{dt} = -\vec{E}$$

First London equation:

$$\boxed{\frac{d\vec{j}}{dt} = \frac{n_s e^2}{m_e}\vec{E}}$$

Classical derivation:

$$-e\vec{E} = m\frac{d\vec{v}}{dt} = -\frac{m}{n_s e}\frac{d\vec{j}}{dt}$$

$$\frac{d\vec{j}}{dt} = \frac{n_s e^2}{m_e}\vec{E}$$

2nd London equation

$$\vec{j} = \frac{-n_s e^2}{m_e} \vec{A}$$

$$\nabla \times \vec{j} = \frac{-n_s e^2}{m_e} \nabla \times \vec{A}$$

Second London equation:

$$\boxed{\nabla \times \vec{j} = \frac{-n_s e^2}{m_e} \vec{B}}$$

London equations

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m}(-i\hbar\nabla - qA)^2\psi + V\psi$$

+ cooper pairs condense into the same state

First London equation:

$$\frac{d\vec{j}}{dt} = \frac{n_s e^2}{m_e}\vec{E}$$

Second London equation:

$$\nabla \times \vec{j} = \frac{-n_s e^2}{m_e}\vec{B}$$

Meissner effect

Combine second London equation with Ampere's law

$$\nabla \times \vec{j} = \frac{-n_s e^2}{m_e} \vec{B} \qquad \nabla \times \vec{B} = \mu_0 \vec{j}$$

$$\nabla \times \nabla \times \vec{B} = \frac{-n_s e^2 \mu_0}{m_e} \vec{B}$$

$$\nabla \times \nabla \times \vec{B} = \nabla \left(\nabla \cdot \vec{B} \right) - \nabla^2 \vec{B}$$

Helmholtz equation: $\lambda^2 \nabla^2 \vec{B} = \vec{B}$

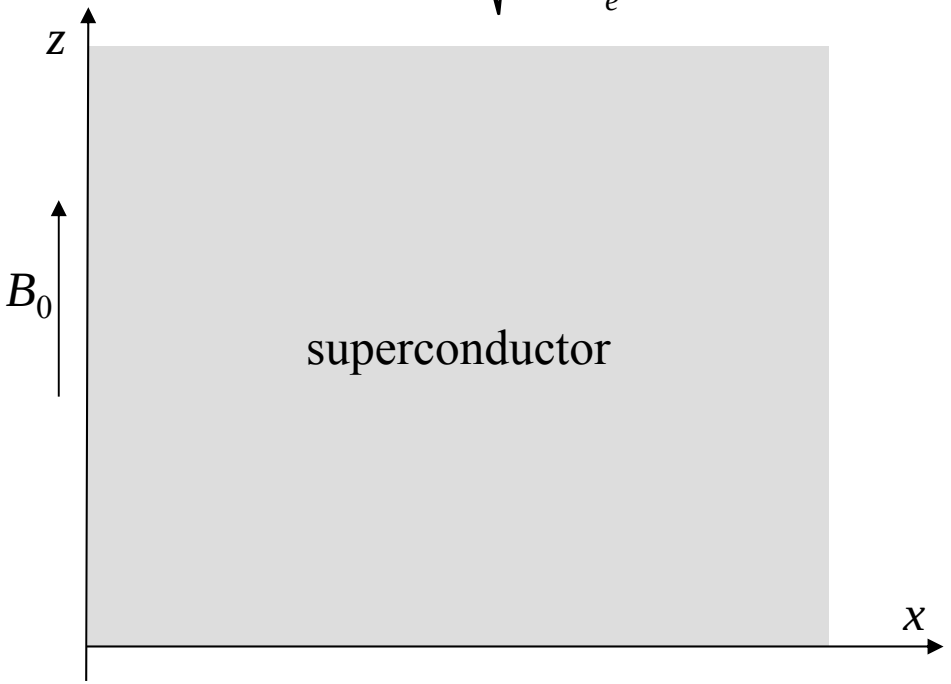
London penetration depth: $\lambda = \sqrt{\frac{n_s e^2 \mu_0}{m_e}}$

Meissner effect

$$\lambda^2 \nabla^2 \vec{B} = \vec{B} \qquad \lambda = \sqrt{\frac{n_s e^2 \mu_0}{m_e}}$$

solution to Helmholtz equation:

$$\vec{B} = \vec{B}_0 \exp\left(\frac{-x}{\lambda}\right)\hat{z}$$



Al	λ = 50 nm
In	λ = 65 nm
Sn	λ = 50 nm
Pb	λ = 40 nm
Nb	λ = 85 nm

$$\nabla \times \vec{B} = \mu_0 \vec{j} \qquad \vec{j} = \frac{\vec{B}_0}{\mu_0 \lambda} \exp\left(\frac{-x}{\lambda}\right)\hat{y}$$

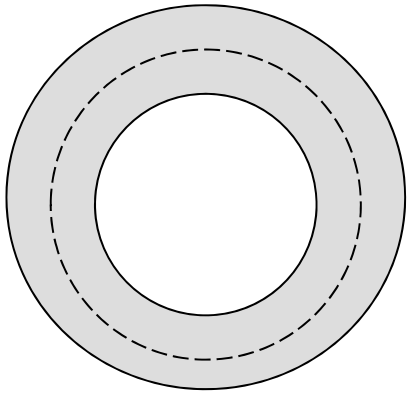
Flux quantization

$$\vec{j} = \frac{-e\hbar n_{cp}}{m_e}\left(\nabla\theta + \frac{2e}{\hbar}\vec{A}\right)$$

For a ring much thicker than the penetration depth, $j = 0$ along the dotted path.

$$0 = \left(\nabla\theta + \frac{2e}{\hbar}\vec{A}\right)$$

Integrate once along the dotted path.



$$\oint \nabla\theta \cdot d\vec{l} = 2\pi n = -\frac{2e}{\hbar}\oint \vec{A}\cdot d\vec{l} = -\frac{2e}{\hbar}\int_S \nabla\times\vec{A}\cdot d\vec{s} = -\frac{2e}{\hbar}\int_S \vec{B}\cdot d\vec{s} = -\frac{2e}{\hbar}\Phi$$

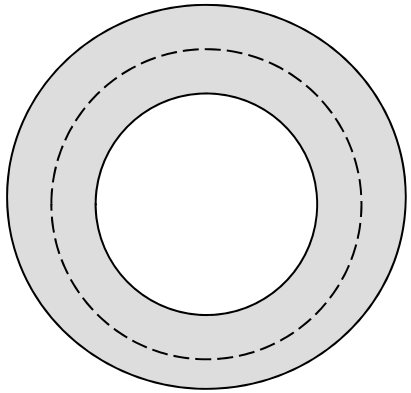
Stokes' theorem

magnetic flux

Flux quantization

$$\oint \nabla\theta \cdot d\vec{l} = 2\pi n = -\frac{2e}{\hbar}\Phi$$

n = ... -2, -1, 0, 1, 2 ...



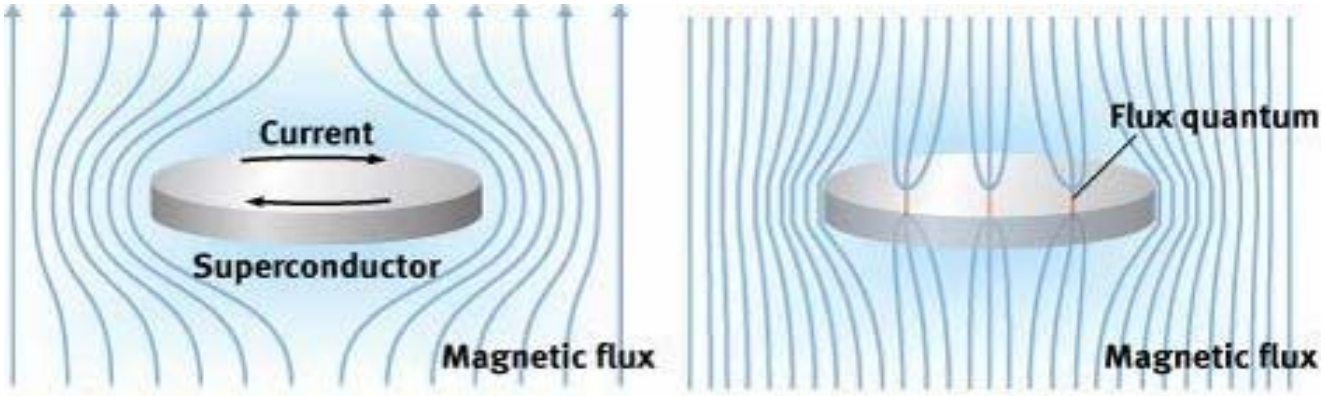
$$2\pi n = \frac{2e}{\hbar}\Phi = \frac{\Phi}{\Phi_0}$$

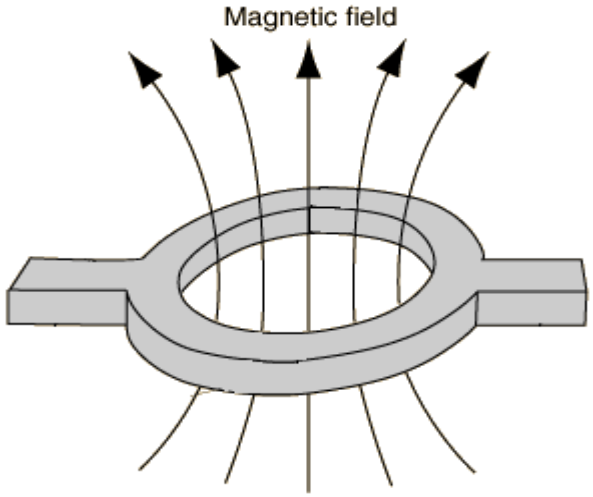
Flux quantization: $\boxed{\Phi = n\Phi_0}$

$$\Phi_0 = \frac{h}{2e} = 2.0679\times10^{-15}\ \ [\mathrm{W} = \mathrm{T\ m}^2]$$

Superconducting flux quantum

Flux quantization

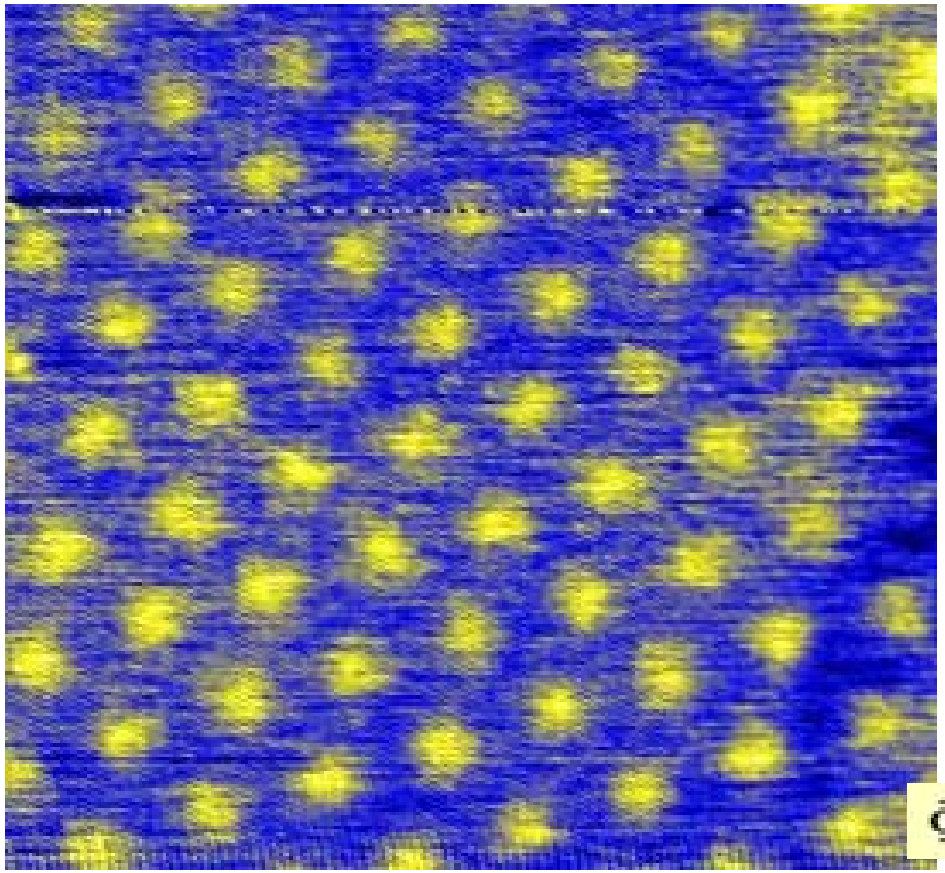


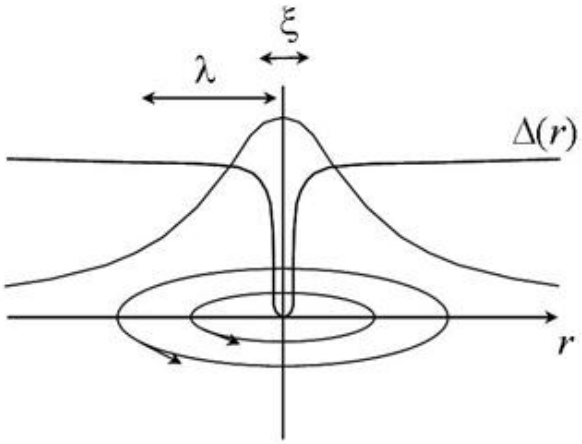


$$\Phi = n\Phi_0 = \frac{nh}{2e}$$

Flux is quantized through a superconducting ring.

Vortices in Superconductors



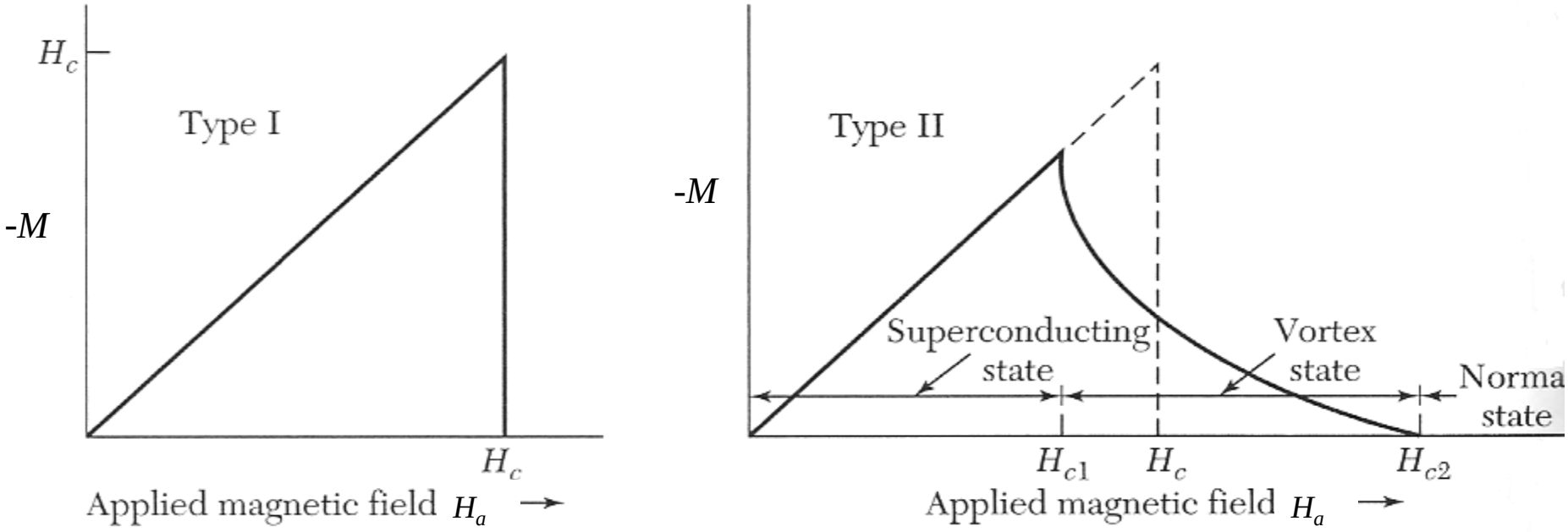


$$\Phi_0 = h/2e \simeq 2 \times 10^{-15} \text{ Tesla m}^2$$

STS image of the vortex lattice in $NbSe_2$.
(630 nm x 500 nm, B = .4 Tesla, T = 4 K)

http://www.insp.upmc.fr/axe1/Dispositifs%20quantiques/AxeI2_more/VORTICES/vortexHD.htm

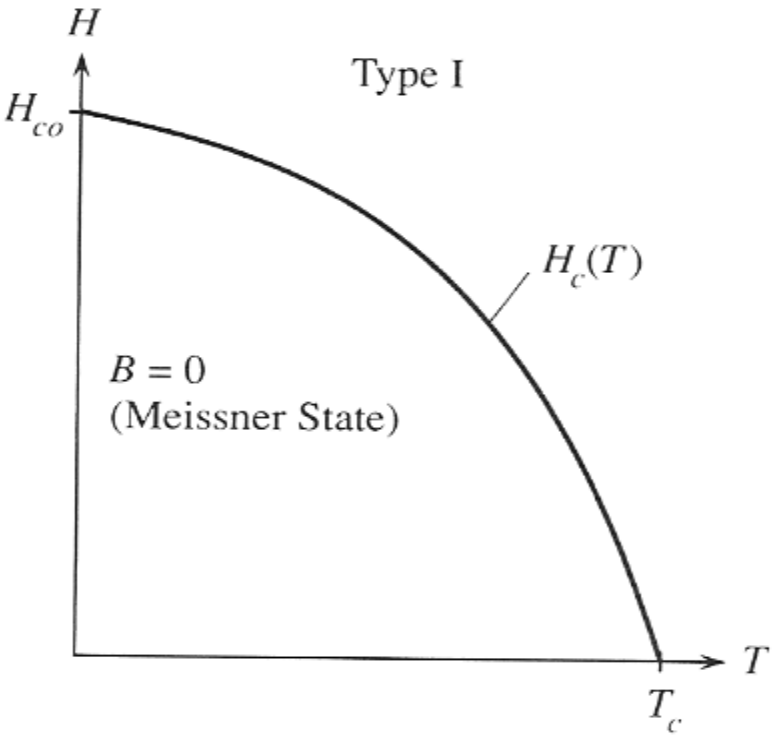
Type I and Type II

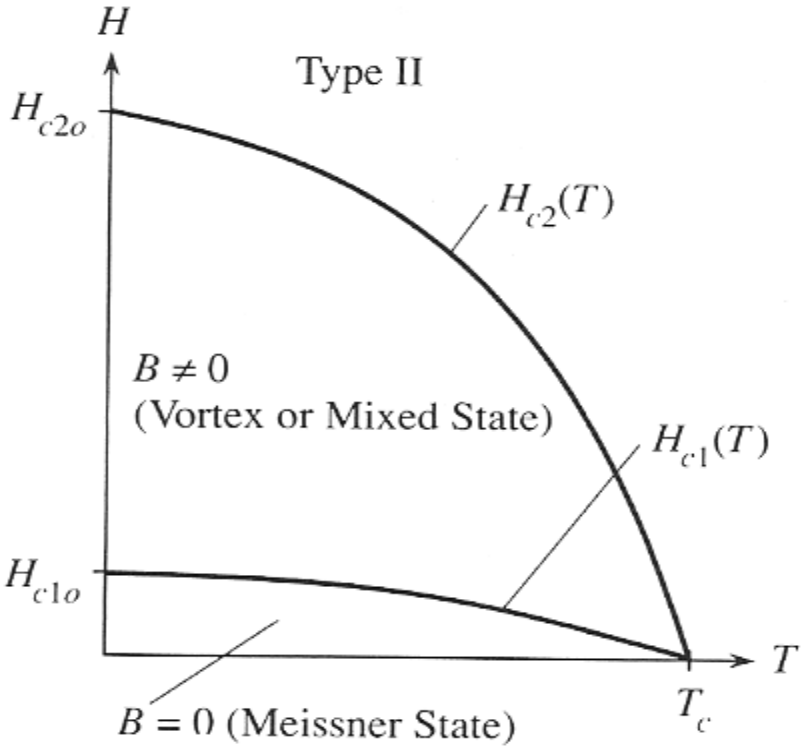


$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right)$$

Superconductors are perfect diamagnets at low fields.
B=0 inside a bulk superconductor.

Type I and Type II





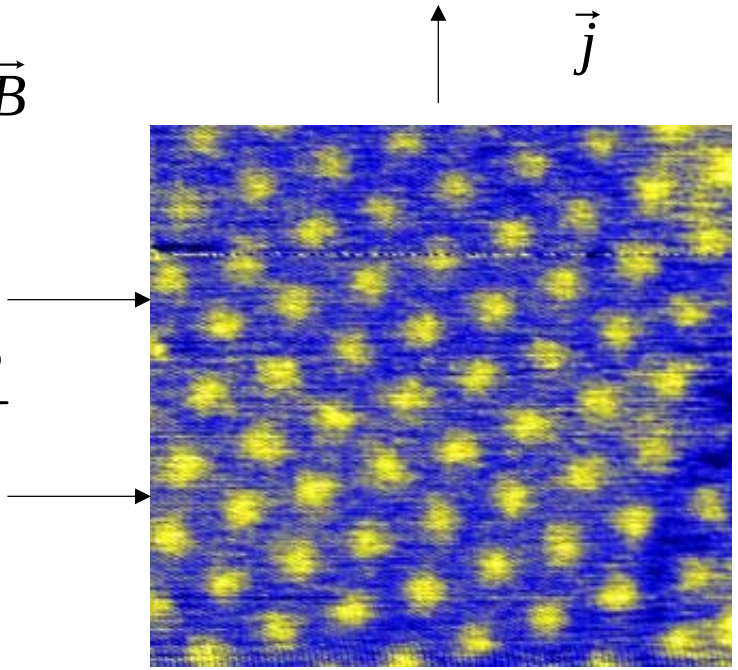
Vortices in Superconductors

Lorentz force $\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$

$\vec{j} = nq\vec{v}$

$\vec{F} = \frac{1}{n}\vec{j} \times \vec{B}$

Faraday's law $V = -\frac{d\Phi}{dt}$



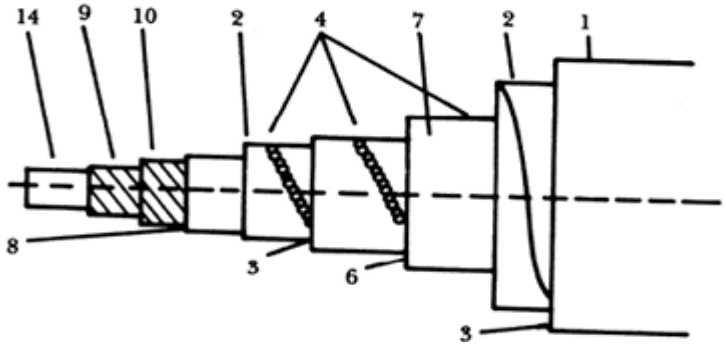
Defects are used to pin the vortices

Superconducting Magnets

Whole body MRI

Magnets and cables

Maglev trains



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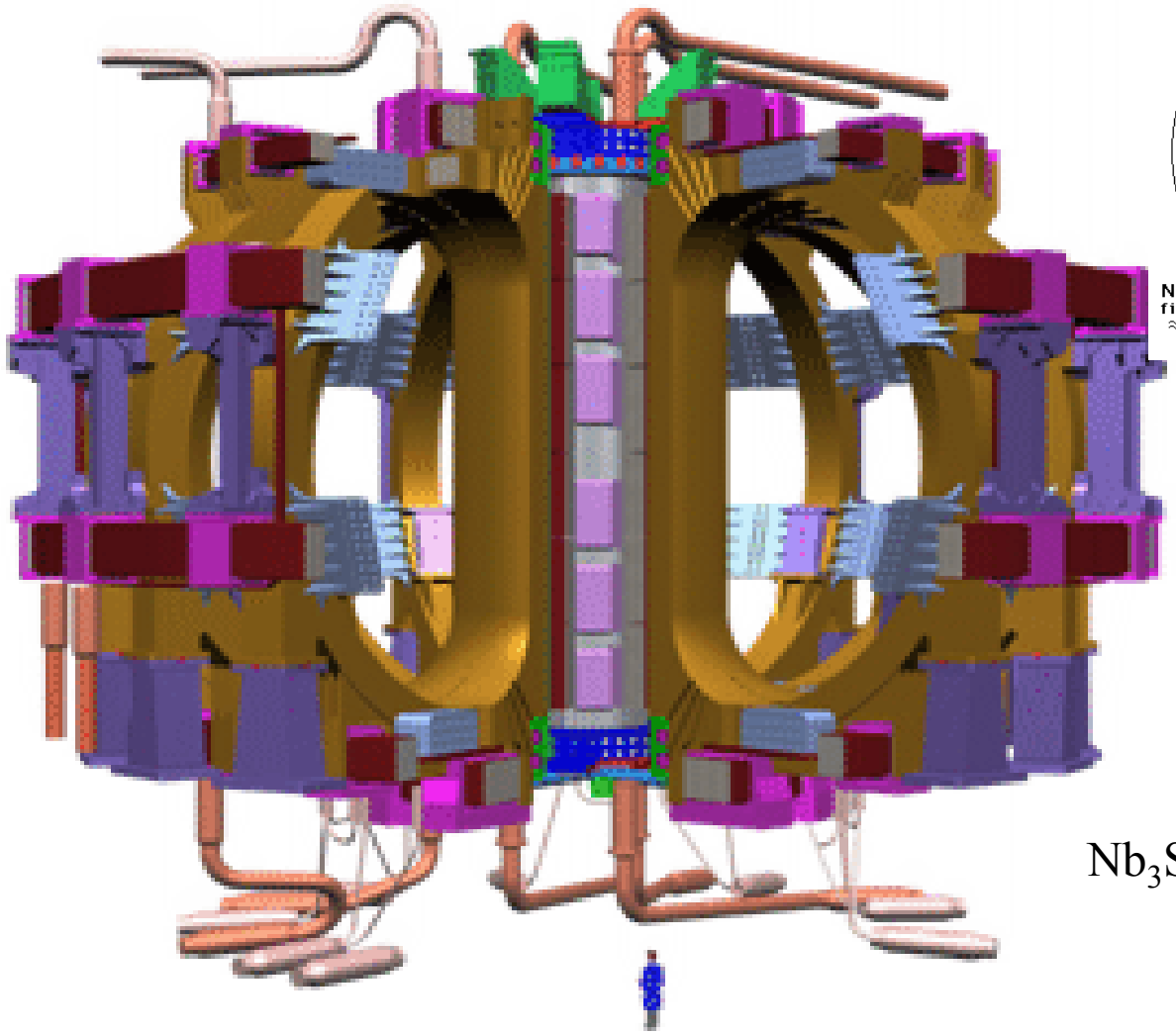
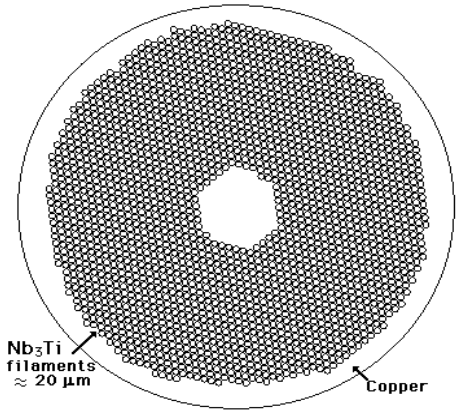


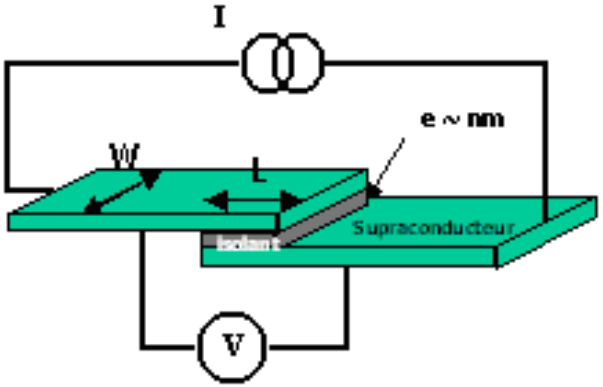


Magnet wire

Nb_3Sn Magnet

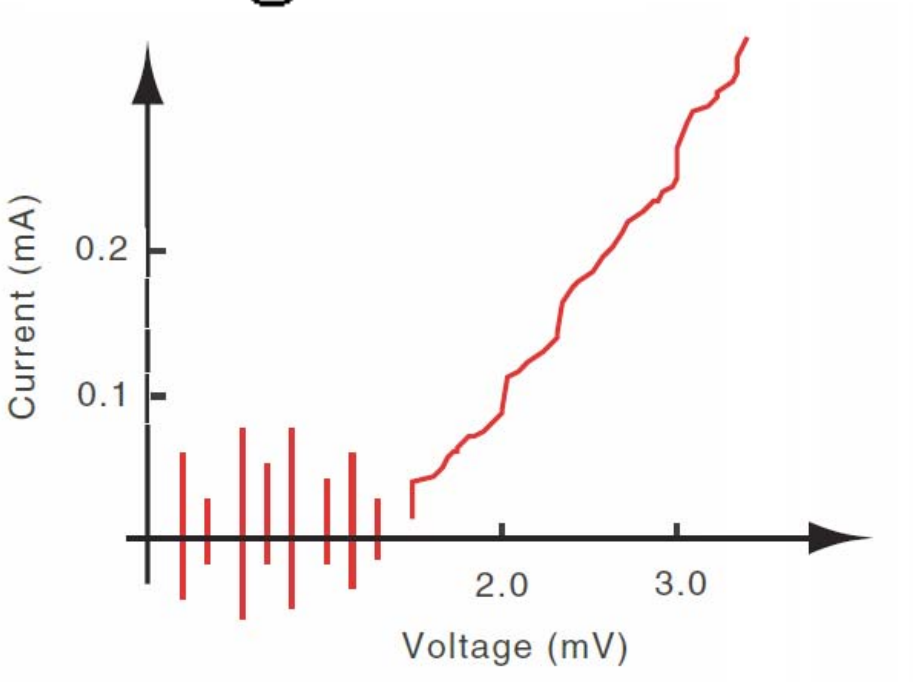
http://www.lne.eu/en/r_and_d/electrical_metrology/josephson-effect-ej.asp

ac - Josephson effect

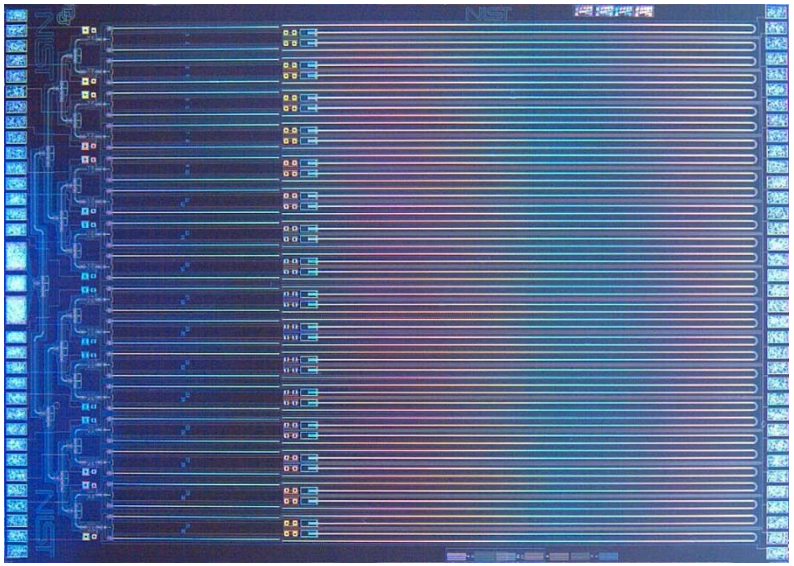


$$V = -\frac{d\Phi}{dt} = n\Phi_0 f = \frac{nhf}{2e}$$

Brian Josephson



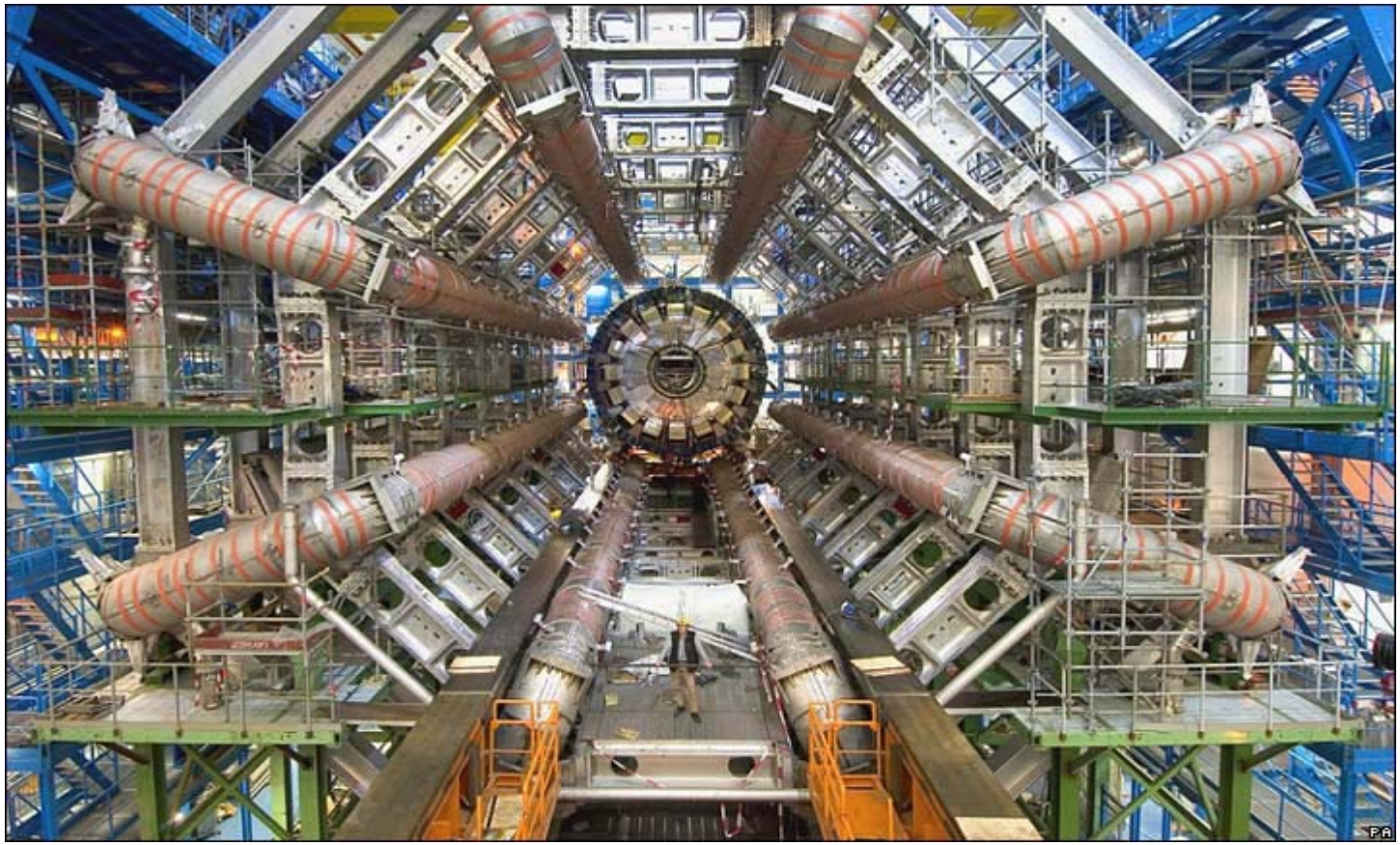
DOI: 10.1140/epjst/e2009-01050-6



10 V standard

http://www.nist.gov/pml/history-volt/superconductivity_2000s.cfm

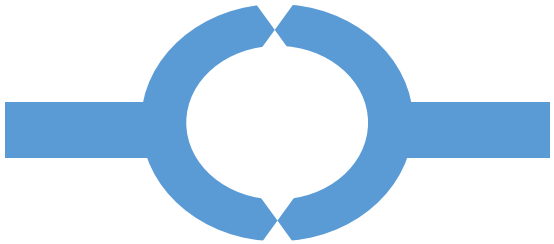
Superconducting magnets



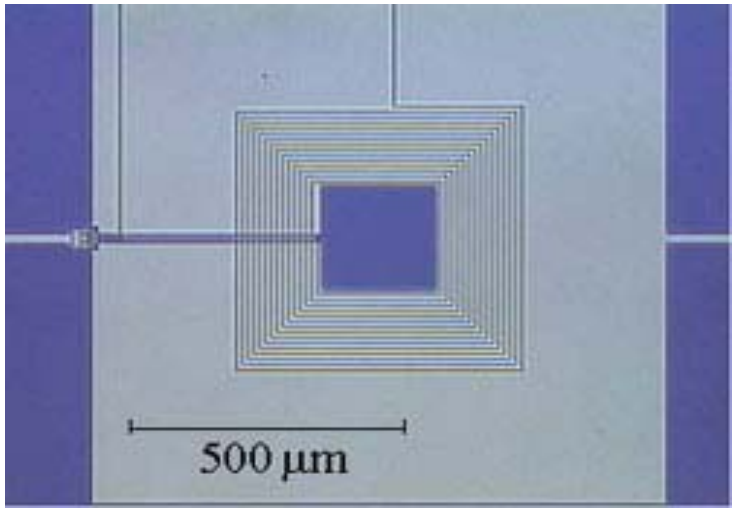
Largest superconducting magnet, CERN
21000 Amps

SQUID

Superconducting quantum interference device



$10^{-6}\ \Phi_0 / (\mathrm{Hz})^{1/2}$



Sensitive detectors

10^{-20} m/ $(\mathrm{Hz})^{1/2}$

Gravity wave detector

Classical linear response theory

Impulse response functions (Green's functions)
Generalized susceptibility
Causality
Kramers-Kronig relations
Fluctuation - dissipation theorem
Dielectric function
Optical properties of solids

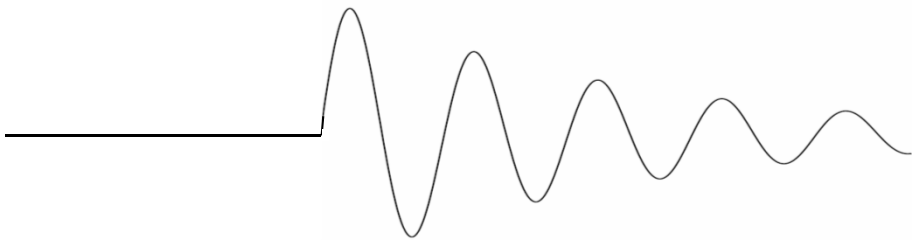
Impulse response function (Green's functions)

A Green's function is the solution to a linear differential equation for a δ-function driving force

For instance, $$m\frac{d^2g}{dt^2}+b\frac{dg}{dt}+kg=\delta(t)$$

has the solution

$$g(t)=\frac{1}{m}\exp\left(\frac{-bt}{2m}\right)\sin\left(\frac{\sqrt{4mk-b^2}}{2m}t\right) \qquad t>0$$



Green's functions

A driving force *f* can be thought of a being built up of many delta functions after each other.

$$f(t) = \int \delta(t - t') f(t') dt'$$

By superposition, the response to this driving function is superposition,

$$u(t) = \int g(t - t') f(t') dt'$$

For instance, $$m\frac{d^2u}{dt^2} + b\frac{du}{dt} + ku = f(t)$$

has the solution

$$u(t) = \int_{-\infty}^{\infty} \frac{1}{m} \exp\left(\frac{-b(t-t')}{2m}\right) \sin\left(\frac{\sqrt{4mk - b^2}}{2m}(t-t')\right) f(t') dt'$$

Green's function converts a differential equation into an integral equation

Generalized susceptibility

A driving function *f* causes a response *u*

If the driving force is sinusoidal,

$$f(t) = F_0 e^{i\omega t}$$

The response will also be sinusoidal.

$$u(t) = \int g(t-t')\, f(t')dt' = \int g(t-t') F_0 e^{i\omega t'} dt'$$

The generalized susceptibility at frequency ω is

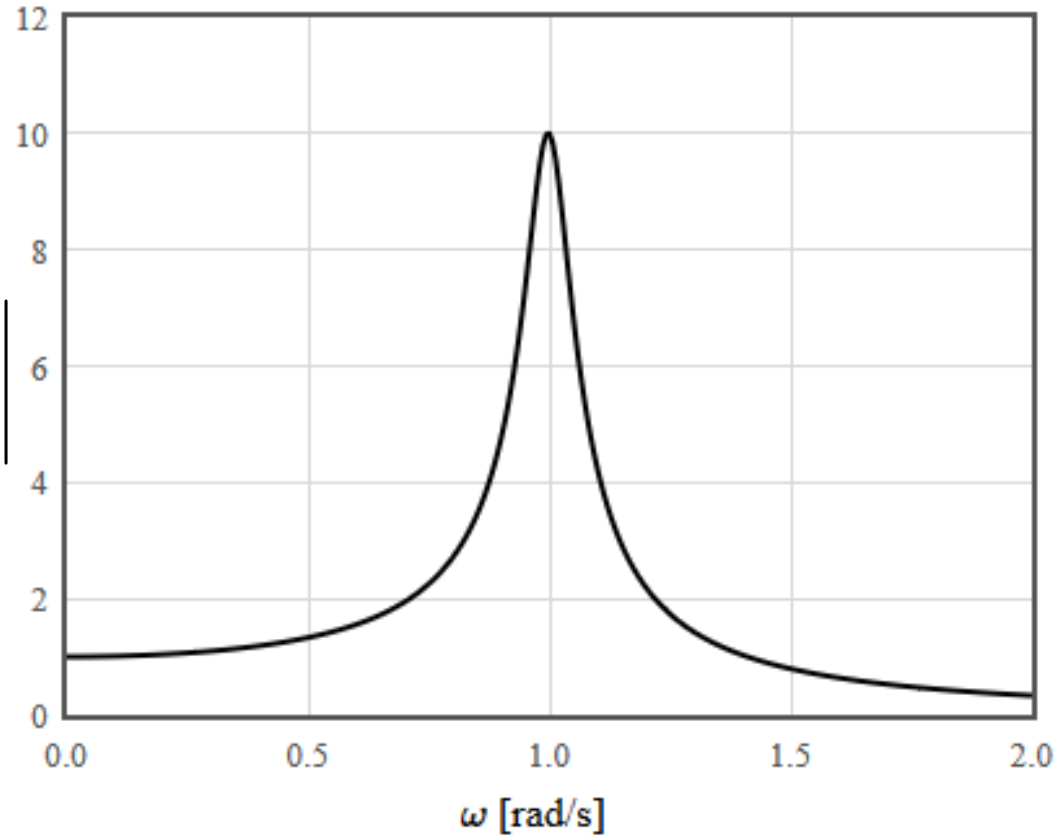
$$\chi(\omega) = \frac{u}{f} = \frac{\int g(t-t') e^{i\omega t'} dt'}{e^{i\omega t}}$$

Generalized susceptibility

$m =$ 1 [kg] $b =$ 0.1 [N s/m] $k =$ 1 [N/m] Plot

$Q = \frac{\sqrt{mk}}{b} =$ 10

$$|\chi(\omega)| = \left|\frac{u}{f}\right|$$



Generalized susceptibility

$$\chi(\omega) = \frac{u}{f} = \frac{\int g(t-t')e^{i\omega t'}dt'}{e^{i\omega t}}$$

Since the integral is over t', the factor with t can be put in the integral.

$$\chi(\omega) = \int g(t-t')e^{-i\omega(t-t')}dt'$$

Change variables to $\tau = t - t'$, $d\tau = -dt'$, reverse the limits of integration

$$\chi(\omega) = \int g(\tau)e^{-i\omega\tau}d\tau \qquad F_{1,-1}$$

The susceptibility is the Fourier transform of the Green's function.

$$g(t) = \frac{1}{2\pi}\int \chi(\omega)e^{i\omega t}d\omega$$

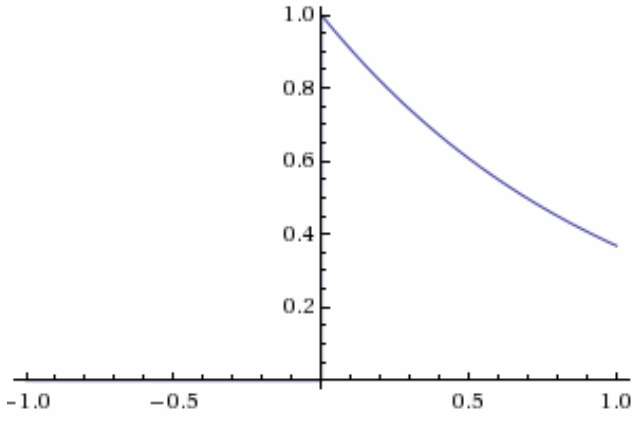
First order differential equation

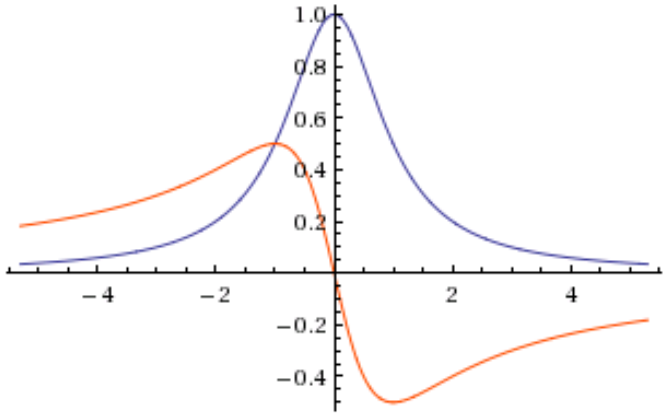
$$\chi(\omega) = \int g(t) e^{-i\omega t} dt$$

$$m\frac{dg}{dt} + bg = \delta(t)$$

$$g(t) = \frac{1}{m} H(t) \exp\left(-\frac{bt}{m}\right) \qquad \frac{b}{m} > 0$$

$$\chi(\omega) = \frac{1}{m} \frac{\frac{b}{m} - i\omega}{\left(\frac{b}{m}\right)^2 + \omega^2}$$





The Fourier transform of a decaying exponential is a Lorentzian

Susceptibility

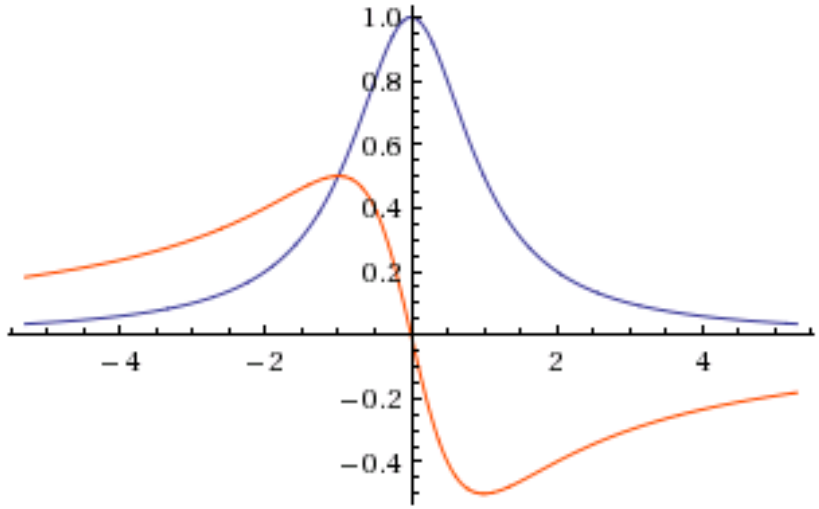
$$m\frac{du}{dt} + bu = F(t)$$

Assume that u and F are sinusoidal $\quad u = Ae^{i\omega t} \quad F = F_0 e^{i\omega t}$

$$i\omega mA + bA = F_0$$

$$A = \frac{F_0}{b + i\omega m} = F_0 \frac{b - i\omega m}{b^2 + m^2\omega^2}$$

$$\chi = \frac{u}{F} = \frac{1}{m}\frac{\frac{b}{m} - i\omega}{\left(\frac{b}{m}\right)^2 + \omega^2}$$



The sign of the imaginary part depends on whether you use $e^{i\omega t}$ or $e^{-i\omega t}$.